PROFESSOR: Today, we continue with our discussion of WKB. So a few matters regarding the WKB were explained in the last few segments. We discussed there would be useful to define a position-dependent momentum for a particle that's moving in a potential. That was a completely classical notion, but helped our terminology in solving the Schrodinger equation and set up the stage for some definitions. For example, a position-dependent de Broglie wavelength that would be given in terms of the position-dependent momentum by the classic formula that de Broglie used for particles that move with constant momentum.

Then in this language, the time independent Schrodinger equation took the form p squared on psi is equal to p squared of x times psi. This is psi of x. And we mentioned that it was kind of nice that the momentum operator ended up sort of in the style of an eigenvalue. The eigenvalue p of x.

We then spoke about wave functions that in the WKB approximation would take a form of an exponential with a phase and a magnitude-- so the usual notation we have for complex numbers. So the psi of x and p would be written as rho of x and t e to the i over h bar s of x and t. That's a typical form of a WKB wave function that you will see soon.

And for such wave functions, it's kind of manifest that rho here is the charge density. Because if you take the norm squared of psi, that gives you exactly rho. The phase cancels. On the other hand, the computation of the current was a little more interesting. And it gave you rho times gradient of s over m-- the mass of the particle. So we identified the current as perpendicular to the surfaces of constant s or constant phase in the exponent of the WKB equation.

Our last comment had to do with lambda. And we've said that we suspect that the semiclassical approximation is valid in some way when lambda is small compared to a physical length of a system or when lambda changes slowly as a function of position. And those things we have not quite yet determined precisely how they go.

And some of what you have to do today is understand more concretely the nature of the approximation. So the semiclassical approximation has something to do with lambda slowly varying and with lambda small, in some sense. And since you have an h bar in here that would make it small if h bar is small, we also mentioned we would end up considering the limit as a sort of imaginary or fictitious limit in which h bar goes to 0.

So it's time to try to really do the approximation. Let let's try to write something and approximate the solution. Now, we had a nice instructive form of the wave function there, but I will take a simpler form in which the wave function will be just a pure exponential.

So setting the approximation scheme-- so approximation-- scheme. So before I had the wave function that had a norm and a phase. Now, I want the wave function that looks like it just has a phase. You would say, of course, that's impossible. So we will for all the time independent Schrodinger equation.

So we will use an s of x. And we will write the psi of x in the form e to the i h bar s of x. And you say, no, that's not true. My usual wave functions are more than just phases. They're not just faces. We've seen wave functions have different magnitudes. Here that wave function will have always density equal to 1.

But that is voided-- that criticism is voided-- if you just simply say that s now may be a complex number. So if s is itself a complex number, the imaginary part of s provides the norm of the wave function. So it's possible to do that. Any rho up here can be written as the exponential of the log of something and then can be brought in the exponent. And there's no loss of generality in writing something like that if s is complex.

Now, we have the Schrodinger equation. And the Schrodinger equation was written there. So it's minus h squared d second d x squared of psi of x. It's equal to p squared of x e to the i over h bar s.

Now, when we differentiate an exponential, we differentiate it two times. We will have a couple of things. We can differentiate the first time-- brings an s prime down. And the second time you can differentiate the exponent or you can differentiate what is down already. So it's two terms. The first one would be-- imagine differentiating the first one. The s goes down and then the derivative acts on the s. So you get i over h bar s double prime. Let's use prime notation.

And the other one is when you differentiate the ones here. It brings the factor down, again, there. So it's plus i over h bar s prime squared. And then the phase is still there, and it can cancel between the left and the right. So this is equal to p squared of x.

So I took the derivative and cancelled the exponentials. So cleaning this up a little bit, we'll have this term over here. The h's cancel, the sine cancels, and you get s prime of x squared minus i h bar s double prime. It's equal to p squared of x.

OK. Simple enough. We have a derivation of that equation. And the first thing that you say is, it looks like we've gone backwards. We've gone from a reasonably simple equation, the Schrodinger equation-- second order, linear differential equation-- to a nonlinear equation. Here is the function squared, the derivative squared, second derivative, no-- there's nothing linear about this equation in s. If you have one solution for an s and another solution, you cannot add them.

So this happens because we put everything in the exponential. When you take double derivatives of an exponential you get a term with double derivatives and a term with a derivative squared. There's nothing you can do. And this still represents progress in some way, even though it has become an equation that looks more difficult. It can be tracked in some other way.

So the first and most important thing we want to say about this equation is that it's a nonlinear differential equation. The h bar term appears in just one position here. And let's consider a following claim-- I will claim that i h bar s double prime-- this term-- is small when v of x is slowly varying. You see, we're having in mind the situation with which a particle is moving in a potential-- a quantum particle.

So there it s, b of x is well-defined. That's a term that goes into this equation. So this is partly known. You may not know the energy, but the potential you know. And my claim-- and perhaps a very important claim about this equation that sets you going clearly-- is that when v of s is slowly varying, this term is almost irrelevant. That's the first thing we want to make sure we understand.

So let's take v of x is equal to v0 at constant. So this is the extreme case of a slowly varying potential. It just doesn't vary at all. In that case, p of x is going to be a constant. And that constant is the square root of 2m e minus V0.

And what do we have here? We have a free particle. This v of x is a constant. So the solution of the Schrodinger equation, that you know in general, is that psi of x is e to the i p0 x over h bar. That solves the Schrodinger equation.

Now, we're talking about this equation. So to connect to that equation in this situation of constant potential constant, momentum in the classical sense, and a free particle with that constant momentum, remember that s is a term here in the exponential. So for this solution here, s of x is equal to p0 x. That's all it is. It's whatever is left when you single out the i over h bar.

And let's look at that thing. That should be a solution. We've constructed the solution of the momentum. equation for constant potential. We've read what s of x is. That should solve this equation.

And how does it manage to solve it? It manages to solve it because s prime of x is equal to p0. s double prime of x is equal to 0. And the equation works out with the first term squared-- p0 squared-- equal to p of x squared-- which is p0 squared. So this term, first term in the left-hand side, is equal to the right-hand side This term is identically 0. So when the potential is constant, that term, i h bar s double prime, plays no role, It's 0.

So the term i h bar s double prime is equal to 0. So the claim now follows from a fairly intuitive result. If the potential is constant, that term in the solution is 0. If the potential will be extremely, slowly varying, that term should be very small. You cannot expect that the constant potential has a solution. And you now do infinitesimal variation of your potential, and suddenly this term becomes very big.

So for constant v, i h bar is double prime equal 0. So for slowly varying v of x i h bar is double prime, it should be small in the sense that this solution is approximately correct. So if we do say that, we've identified the term in the equation that is small when potentials are slowly varying. Therefore, we will take that term as being the small term in that equation. And this will be nicely implemented by considering, as we said, h bar going to 0, or h as a small parameter.

We will learn, as we do the approximation, how to quantify what something that we call slowly varying is. But we will take h now as a small parameter-- that makes that term small-- and setup an expansion to solve this equation. That is our goal now.

So how do we set it up? We set it up like this-- we say s of x, as you've learned in perturbation theory, is s0 of x plus h bar s1 of x-- the first correction-- plus h bar squared is s2 of x and higher order. Now, s already has units of h bar-- so s0 will have units of h bar too. s1, for example, has no units. So each term has a different set of units.

And that's OK, because h bar has units. And this will go like 1 over h bar units and so forth and so on. So we have an expansion. And this we'll call our semiclassical expansion. As we apply now this expansion to that equation, we should treat h as we treated lambda in perturbation theory. That is, we imagine that this must be hauled order by order in h bar, because we imagine we have the flexibility of changing h bar.

So let's do this. What do we have in the equation? We have s prime. So the equation has s0 prime plus h bar s1 prime. Now, I will keep terms up to order h bar. So I will stop here-- plus order h bar squared minus i h bar s double prime. So that should be s0 double prime plus order h bar equal b squared of x.

So let's organize our terms. We have s0 prime primed squared on the left-hand side. Minus p squared of x-- I bring the p to the left-hand side-- plus h bar. So these are terms that have no h bar.

And when we look at the h bar from the first term there, we square this thing. So you get the cross product between the s0 prime and the s1 prime. So you get 2-- the h prime already is there-- s0 prime s1 prime. And from the second term, you get minus i s0 double prime plus order h squared equals 0.

So I just collected all the terms there. So if we're believing in this expansion, the first thing we should say is that each coefficient in the power series of h bar is 0. So we'll get two equations. First one is s0 prime squared is equal to p squared of x. That's one. That's this term equal to 0.

And here, we get-- let's write it in a way that we solve for the unknown. Supposedly from this first equation we, can solve for s0. If you know s0, what do you want now to know is s1 prime. So let's write this as s1 prime is equal to i s0 double prime over 2 s0 zero prime. So these are my two equations.