PROFESSOR: OK. So let's try to solve this. So my classical approximation is about solving these equations. So let's see what we get. Well, the first equation is kind of simple.

I think everybody has the temptation there to just take the square root, and that's what we should do, s0 prime is equal to plus minus p of x. And therefore s0 of x is equal to plus/minus the integral up to x of p of x prime dx prime.

You see p of x is pretty much known. If you know the energy of your particle, then it's completely known, and it depends on e minus v. So this is a solution in terms of p of x. We should think of solving the differential equation in terms of p of x.

Now, as a first order differential equation, there's a constant of integration. And we'll pick it up to be a number here, x0. So we start integrating from some place. If you integrate from another place, you're shifting the constant of integration. The main thing is that the x derivative here acts as the upper limit and gives you the p of x of that equation.

So this is our solution. Even the plus minus should not disturb us. If you have the p squared, you don't know if the particle is moving to the left or to the right. So that ambiguity is perfectly reasonable. Particle can be moving to the left or to the right. Now, we look at the second equation.

So s1 prime is equal to i over 2. s0 double prime, if you have s0 double prime, you have here s0 prime, so you take another derivative. So that's a plus minus p prime of x divided by s0 prime, which is plus minus p of x. That's kind of nice. The sine is going to cancel.

So we have here i over 2 p prime of x over p of x or i over 2 logarithm of p of x prime. The derivative of the logarithm is 1 over the function, and then by chain rule, you get the p prime there. So if s1, the prime derivative is the derivative of this thing, so s1 is going to be i over 2 log of p of x plus a constant.

So let's reconstruct our solution. That's not hard. We wrote the answers up there. So the wave function is e to the i over h bar times s, and s is what we had there. So our wave function is e to the i over h bar s, and s was s0 plus i plus h bar is 1.

There's more. Is it right? But we're going to ignore it. We didn't go that far. In fact, nobody goes higher in the WKB approximation. So what do we have here? I'll write it. This term is kind of interesting. We have e to the i h bar of x times e to the i s1. And s1 was i over 2 log of p of x plus a constant.

So look at this items i is minus 1, 2. So you have 1/2. This becomes e to the minus 1/2 logarithm of b of x. And 1/2 the logarithm of b of x is e to the minus log of square root of b of x, and when you go like that e to minus that is 1 over the function. So p of x like that, and then we have e to the i over h bar integral from x0 to x p of x prime dx prime. This is the classic WKB approximation, classic result.

So as promised, this is of the form of a scale factor here, a rho, the square root of rho times a phase. So we did begin with what looked like a pure phase, but then we said s of x is complicit in fact. s0 was real, but s1 was imaginary. With s1 imaginary, the rho of s1 was to provide the magnitude.

And this is an intuition, this approximation scheme. The first thing you have to get right is the phase. Once you get the phase right, the next order, you get the amplitude of the wave right. That comes to second order. That's a next effect. So this is our solution.

When I began today, I reminded you that we have WKB solutions of the form square root of rho e to the is, and we calculated some things for that. So because of the signs, s0, I dropped the sign, this plus or minus. Let me write the general solutions of WKB slightly more complete.

Let's be more complete. It's important to see the whole freedom here. So if we have e greater than v, remember when e is greater than v, the p of x is a real quantity. And we wrote, and we said that p of x we would write as h bar k of x.

You know, I think I should have probably, for convenience here, let's put the constant. We're not attempting to normalize these wave functions. We could not attempt to do it, because we don't know what p of x is. And this function may have limited validity as we've spoken. But I had the constants here. This constant could have a real or imaginary part. It would affect this a. So let's put it there.

OK. So if p of x is hk of x, we can have the following solutions, psi of x and t equal a, another constant, square root of p-- let's go simpler-- square root of k. It's a different a. And here, e to the i, since p is h bar, it cancels here. So we have a simpler integral as well, x0 to x, k of x prime, dx prime.

So that's that term. I just use the opportunity to replace p for k, which simplifies your life, simplifies all these constants. So the other solution is the wave moving in a different direction. So k of x, e to the minus i x0 to x, k of x prime v of x prime. So that is your solution when you have e greater than v.

If we have e less than v, we still have a solution, and we said that p of x, in that case, would be equal to i h bar kappa of x. We use that notation. If e is less than v, this is a negative number. So p of x is i times some positive number and square root of a positive number. And we called it kappa last time.

So this is the letter we usually use for spatial dependence in regions where the wave function decays exponentially, which is what it's going to happen here. So what is the psi of x and t is equal to a constant c over square root of kappa of x e to the-- the i will disappear, and there will be two solutions, one with plus, one with minus.

That's the reason I don't have to be very careful in saying whether this is i or minus i. There's, anyway, two solutions. At this stage, we don't need to worry. So this is from x0 to x, kappa of x prime, dx prime, plus d over square root of kappa of x, e to the minus x0 to x, kappa of x prime, dx prime.

So this is the complete solution of WKB. If you are in the classically allowed region top or in the classically forbidden region, it's important to realize that this function, the second term is the decaying exponential. As x increases, the integral accumulates more and more value, and the wave function gets more and more suppressed.

This is a growing kind of exponential. In the previous iteration in your life, kappa was a constant, if you had constant potentials. And this would be e to the kappa x basically. But here, you must think of kappa being some positive number, positive function, as you integrate and x grows, your cube-- the integral becomes bigger. And this is a growing exponential.

So the sign tells you that, especially because we've ordered the limits properly. So we have a decaying and growing exponential. At this moment, we're pretty much done with what WKB does for you. Although, we have a few things still to say.

So this will be in terms of comments about the general validity of such approximation, but first even some comments about the current and charge density. So let's consider this equation I. Let's just make the comments, comments for equation I on one.

What is the charge density or the probability density in this case, rho would be psi squared, and in case one, is equal to a squared over k of x? You could, if you wish, this is a perfectly nice formula, multiply it by h bar up and down. And that's the momentum.

So it's h bar a squared over p of x. And you could say this is h bar over m a squared over v of x, a local velocity, p of x is m over a local velocity. And this is an intuition you've had for a long time. The probability density or the amplitude of the wave is going to become bigger in the regions where the particle has smaller velocity.

That's the regions of the potential where the particle spends more time, and it's an intuition that almost immediately comes here. This k is essentially the momentum. So that's essentially the square root of the velocity. And this coefficient, therefore, becomes bigger, as the velocity is smaller.

That's part of the intuition you've had for a long time, regarding these quantities. The other piece is the computation of the current from this equation I. Remember, the current is h bar over m times the imaginary part of psi star gradient psi. So it's a long computation. But we did it for the case we had before.

We said that the current is rho times gradient of s over m for the case when the wave function is written in rho e to the s form. So in here, rho is already determined is a squared over k of x. We have the 1 over m, and the gradient of s-- s is this quantity, e to the i h bar times s.

So the gradient of s is just p of x, so h bar k of x. Now, they cancel. And it's very fast. They cancel. Would have been a major disaster if they didn't. This is a number. It's a squared over m.

The reason it cancels is that it would have failed the conservation law otherwise. The rho dt plus the divergence of j should be 0. In our case, rho has no time dependence. The wave function that we are considering, our time independent Schrodinger equations, we're considering energy eigenstates. And the current must be a constant. In an energy eigenstate, the current cannot be a spatially varying constant, because then the current would accumulate in some place, and that's inconsistent with stationary states.

And in fact, this is 0, and the versions of j in this case would be dj dx, and if it would have had some x dependence, it would have destroyed this equation. So dj dx is also 0, and that's all consistent.