PROFESSOR: So let's try to do return to seeing, OK, we solved equation. We seem to be OK. What did we really approximate? We didn't approximate saying h bar goes to 0. We did a more serious physical approximation. And let's try to see what we really did. So I think the whole clue is in this top equation there. You have the first term and the second term. And our claim is that the second term is smaller than the first term with h bar there.

So for example-- now of course, in this solution, the first term is identically 0 and the second term, the coefficient of h bar, is identically 0. But we can look at one of those, for example, and say that-- so the validity of the approximation. Validity of the approximation. It's pretty useful to do this.

So we say, for example there, that term, h bar s0 double prime that enters into the order h bar part of the equation, the absolute value of it must be much smaller than a typical term s0 prime, for example, s0 prime squared, in the first term. So each term-- so basically, I'm saying each term in the first bracket must be much larger than each term in the second bracket. And you could have picked any ones because they're all equal, after all.

So let's see if that is the case. So recall that s0 prime from there is really p of x plus minus p of x. So what do we have here? h bar and s0 double prime is dp vx. Must be much smaller than p of x squared. Now it's a matter of playing with these things a little bit until you find some way that the equality tells you story. And the way I'll do it is by saying that this is h bar 1 over p squared of x dp vx is much smaller than 1.

And here I'll write this as h bar d-- no h bar. ddx of h bar over p. Look what I did. ddx of 1 over p is 1 over p squared dp vx and the h bar I put it there. Here we go. What does this say? This is the local De Broglie wave length. This is saying that ddx of the local the De Broglie wave length must be much smaller than 1. A nice result, your local De Broglie wavelength must have a small derivative.

So this is the physics translation of the semi-classical approximation. h bar going to 0 is a mathematical device, but this is physical. This is telling you what should happen. Most of us look at that and see an easier way to understand that equation. v lambda vx. It has the right units. Lambda has units of length. x has units of length. So that derivative must have no units. And if it's supposed to be small, it should be small compared to 1.

So this is the conventional inequality. Most of us would prefer, maybe, to write it like this-- lambda d lambda dx is much smaller than lambda. And I think this is a little clearer because this is how much the De Broglie wave length changes over a distance equal to the De Broglie wavelength. So you have a De Broglie wave length and the next De Broglie wave length. How much did it change? That must be small compared to the De Broglie wavelength.

So the change of the De Broglie wave length after you move one De Broglie wave length must be smaller than the De Broglie wave length. I don't know if you like it. Otherwise, you can take this one. I'll do another one, another version of the inequalities. And you can play with those inequalities. It kind of takes a while until you convince yourself that you're not missing anything. Think of p squared equal to m e minus v of x.

Take a derivative, vvx. So I'll have p, p prime. This is 2 pp prime. But with this 2, I'm going to cancel it. At some point, of course, we're taking all kinds of factors of 2 and ignoring them. Remember that true De Broglie wave length is h over p not h bar over p. Factors-- by the time you go to this inequality, two pis are gone. m. We're differentiating with respect to x and taking absolute value. So we'll write it like this.

Or vvvx vx equals 1 over m pp prime. That is so far exact. Let me multiply by a lambda. So I'll have a lambda. And a lambda, vv dx, is equal to lambda. OK. Lambda is h bar over p. So I can cancel one of these p's and get h bar over m p prime. OK.

H bar over m p prime. Now, look at this equation. I'm sorry. We'll play with this a little-- like, trial and error. You're trying to move around your inequality. So here we have something-- h bar dp dx is much less than that. So this term is because of this inequality is much smaller than p squared over m. And now we have something nice. I'll write it here. Lambda of x dv dx is much smaller than p squared over 2m.

That's another nice one. I think this one is the-- and this says that the potential must be slowly varying for this to be true because the change in the potential over at the De Broglie wavelength-- dv dx times lambda of x is an estimate for the change of the potential over the De Broglie wavelength-- is much smaller than the kinetic energy of the particle. So that's, again, another thing that makes sense. It's kind of nice.

So this is the wave at the end of the day, this h bar going to zero approximation has become a physical statement. It is a statement of quantities varying slowly because after all, that's what motivated the expansion from the beginning. So let's see if we ever get in trouble with this. So we're trying to solve physical problems of particles and potentials. And most of the times we're interested in bound states or energy eigenstates, at least the simple energy eigenstates are bound states.

So here is a situation. We have a sketch of a situation. We have a v of x-- this is x. This is a. v of x-- and some energy, e. And let's assume we're looking close enough to the point x equals a so that the v of x, however it curves, at that point is roughly straight. That's a reasonable thing to do. So we'll model v of x minus e-- v of x minus e-- as being linear near x equals a. So this is g times x minus a, where g is some positive constant, which is the slope at this point.

So look at x less than a. At this point, you are in the loud region. You're in the region to the left of the point a where your energy is bigger than the potential. And that's perfectly allowed. So here, e minus v of x, which is the negative of that, would be g a minus x. And p is square root of 2 m e minus v of x, so g a minus x. That's p of x.

So that's your position dependent momentum. It's going to go to 0 at that point. And lambda, which is h bar over p, is h bar over square root of 2 mg 1 over square root of a minus x. So take the derivative-- d lambda d x-- take the absolute value of it. That's h bar over square root of 2 mg times 1/2 1 over a minus x to the three halves. We're differentiating with respect to x.

And now you see the trouble, if you had not seen it before, the validity of the semi-classical approximation is taken and requires d lambda dx to be much smaller than 1. And as you approach x equals to a, this grows without bound. It just becomes bigger and bigger. You can choose g to be large and you can choose m to be large. But still, you get closer and closer, you eventually fail. This thing goes to infinity as x goes to a and grows without limit. And the semi-classical approximation crashes.

You know, I would imagine that many people got this far with this in my classical approximation of writing this and that. But this is a tremendous obstacle. Why is it an obstacle? Why can't we just forget about that region because most of the times you dealing with bound states. So you will have a very slowly varying potential here. But if you want to find bound states, you need the fact that there is a forbidden region where the wave function destroys itself. So whatever you can solve for the wave function is slowly varying here. It's not enough because you need to know how it decays.

And therefore, you need to face this corners where the semi-classical approximation fails. Our problem here is that we know how to write a solution here. We probably know how to write the solution here. Those are these ones. But we have no idea how to write them here. So we cannot connect the two solutions. It's a serious difficulty.

So people work hard on that. And I think that is the breakthrough of the construction of this WKBP [INAUDIBLE]. What they did is they solved the equation exactly in this region, assuming a linear potential. They solved it exactly. And then those functions, the [INAUDIBLE] functions, show up. And you know how the [INAUDIBLE] functions behave. So they solve it here. They related it to the solution to the right, related it to the solution on the left. And in that way, even though we don't have to write the solution in this region, we know how a solution on the middle connects to a solution on the right.

This is the subject of the connection formulas in WKB. We will discuss that next time, or we'll go through some of that analysis because it's interesting and fairly non-trivial. But I will mention one of the connection formulas and use it. That's the rest of what we're going to do today.