PROFESSOR: So here it is, connections formula, connection. So we'll take a situation as follows. Here is the point of x equals a. I'll put just a here. Here is the x-axis. And I will imagine that I have a linear potential.

So we have a linear potential here. Why do we imagine a linear potential? It's because the thing we would really kind of want to think about is what is called the turning point, energy, and a potential. There's a turning point.

But it's clear that near enough to the turning point, this is linear. And the problem for the WKB approximation fails, as we discussed, is precisely in this region. Now, I will make one claim that this is really what's going to connect here. These two asymptotic expansions that we've found here are in fact WKB approximations of our calculations.

So our whole procedure here is going to be assuming that we have a linear potential here, and I will take the linear potential seriously. So I will imagine it even goes forever. And what we're going to do is find a way from the turning point. So the energy is going to be here. Here is v of x. x v of x, the energy.

And then we have this region, which is a WKB region. And we have this region over here that is also WKB region. And the region in the middle is not a WKB region. It's a region where all solutions fail, WKB solutions fail. They're not valid in this region.

So let's assume, we have here, therefore, a potential that is linear, and, therefore v of x minus the energy is going to be a number times x minus a, where g is a positive constant. It has some units. v minus e is a linear function that vanishes at x equals a. So this is a nice description of the situation.

And then you could write a WKB solution. So what is your WKB solution? Let's write the WKB solution, psi WKB on the right, and then we'll write a WKB solution on the left, the right and left.

So here is what we. Typical WKB solution on the right. That is a forbidden region. Square root of kappa of x, you would have a decrease in exponential, e to the minus a to x kappa of x prime, d x prime. And then you have b over square root of kappa of x, e to the a to x kappa of x prime, dx prime.

So this is what you've been told is supposed to be your WKB approximation. Decay in exponential, a pre-factor, growing exponential, a pre-factor. And kappa of x is something that you know from this potential. So this you would write for any potential on the right or on the left as well. And here, your kappa of x squared is 2m over h squared, v of x minus e.

That's the definition of kappa squared. You're in the region where v is greater than e, and therefore, this is equal to 2m g over h squared x minus a. So our next step is a little calculation. We should simplify all these quantities here. We should evaluate them, because we have a kappa. You have a linear function. We can do the integrals. We can put what kappa is. So everything can be done here.

So if we do it, what do we get? So evaluate. We get the following. Now, I will introduce some notation to write this. So first, I'll write it, psi write of x WKB equals 1 over square root of eta. That's a symbol I haven't defined yet, u to the 1/4, e to the minus 2/3, u to the 3/2 plus b square root of eta 1 over u to the 1/4, e to the 2/3, u to the 3/2.

OK, that's what you get. You'll recognize the a and the b's. And here is what we need to introduce a couple of variables to make sure we know what every symbol is. And here, they are. u is eta x minus a. And eta is 2m g over h squared to the 1/3.

Just roughly k squared is proportional to u. So k is proportional to u to the 1/2. And therefore, square root of k gives you the u to the 1/4. The integrals also work that way. And you see already here, things that look like the array function. And that will become even clearer soon.

So this is our solution for the WKB expression on the right, but let's do also the WKB expression on the left. The WKB expression on the left would be another wave of this kind, but with cosines or with exponentials that we've done before. So what did it used to look like? It used to look like k of x and integrals of plus minus i integrals of k of x prime dx prime.

That was the way the WKB solution looked like. I'm now trying to write this expression for the WKB solution on the right. Now, I want to make a remark that even though I'm using the linear potential, in general, if your potential, even eventually becomes something different in this region, I'm good.

So I'm moving away from the bad point sufficiently so that the WKB approximation is good. But I don't want to move away so much, so that the potential fails dramatically to be linear. I have to straddle limited region in fact. I don't want to go too far.

So these were your solutions before in a region where you're in the classically allowed region. Now, this form of the solutions will not be practical for us. So I will write them, instead with sines and cosines. It will be better for us to write it with sines and cosines. So let me write it with sines and cosines. Psi left now WKB of u or x.

Sorry. It's going to be c over square root of k of x, and I will write the cosine of integral from x to a. You see, I'm on the left. So it's natural to write integral from x to a. On the other hand, on the right was natural to write integrals from a to x. And that's what we did, cosine from x to a. We'll have here k of x prime vx prime plus d over square root of k of x sine of x to a k of x prime dx prime.

And you could say look, that's good. You just traded the exponentials for sines and cosines, and that's fine. Still, for what we're going to do later, it's convenient to write our WKB approximations still in a little different way. I can add a phase here and a phase there. And that still is the most general solution, because they're both sines and cosines.

And unless I would put a phase that differs pi over 2, and one becomes equal to the other, no. I'm going to shift both by pi over 4, and therefore, these are still different functions, and it's convenient to do that. So this is our onset for the WKB solution.

We have to do the same thing with it here in which we expand and evaluate in terms of and eta in this quantity. So what do we get? We get c over square root of eta u to the 1/4 cosine 2/3 u to the 3/2 minus pi over 4 plus d over square root of eta u to the 1/4 sine of 2/3 u to the 3/2 minus pi over 4.

So this is our psi left WKB of x. OK. We're in good shape. I'm going to raise this up. And now you could say, all right, you've done the WKB. But on the other hand, how about the exact solution? What is the exact solution to this problem? You've written the WKB approximation for the linear potential, the WKB approximation for the linear potential.

Now, what is the exact solution for the problem? Well, your Schrodinger equation is minus h squared over 2n psi double dot of x plus of x minus the energy on psi is equal to 0. Well, we've written this already a couple of times. So this is minus h squared over 2m psi double dot plus g x minus a psi equals 0. That is our v minus e.

And in terms of the u variable, the differential equation, of course, becomes the second psi, the u squared equal u psi. That's why the area function is relevant here. We get back to that equation. That is the equation for the exact linear potential, and those would be the exact solutions.

So our exact solutions are psi being some linear combination of the array function and the other array function. And now let's do one case first before we get into something that is more complicated. Let's take psi to be the solution to be the array function ai of u.

That's your solution now. If that is your solution, you would have this behavior, the behavior we've noted here for this solution. And now you compare and you look at what you got. You say, all right, I have a solution, and I know how it looks rigorously, how it looks to the right and to the left. That's how it looks, and this is the exact solution.

This is the WKB solution, and I know how it looks to the right and how it looks to the left. And I see the correspondence. We see this term in yellow is a solution that this a decaying exponential, and we see it here.

And we have the red term, the oscillations. And we see it also here. So here is that connection condition. In fact, when you derive this asymptotic expansions, you did the true connection condition, because you connected a single function from the left and from the right. Here it was to the right. Here it was to the left. And it's a connection formula because it's the same function.

So here, we see that if we wrote the WKB solution, and we were aiming at maybe this solution, we would have that this term goes, connects with this term. And the coefficients differ by 1/2. That 1/2 has no other explanation than the asymptotic expansions. There's the square root of pi, these things, and there's the two functions.

So if you put the 1/2 is there in the exponential, so you could put the 1/2 here, and 2a, if you wish. So you realize that in order to have a connection 2a would be equal to c. The two coefficients here are related in that way. There's a number here and a number here. It happens to be 2a and c in this case. So 2a is equal to c. So if we choose a equals 1, we would have c equal 2.

So here is a matching condition therefore. We have that a WKB solution that has the c factor with c equals 2, so it would be 2 overs square root of k of x in here times cosine of this x over a k of x prime, dx prime minus pi over 4 is connected-- we'll see more on that-- with 1 over square root of kappa of x, this solution with a equals to 1, what we have here, e to the minus a over x kappa of x prime dx prime.

So that is the first connection condition. We will discuss it further in a couple of minutes, but this is how you connect things. A single solution of the differential equation looks like this term and like that term on the left and on the right. Here is your WKB solution. This general solution looks like this term, and it then should match to something that looks like that if you have the array function. So in general, for not linear potentials, this is the term that gives rise to that, so we take that this term in general is matched to the top term in the blackboard here.