

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Physics Department

Physics 8.07: Electromagnetism II  
Prof. Alan Guth

October 30, 2012

**PROBLEM SET 7 REVISED\***

**DUE DATE:** Friday, November 2, 2012. Either hand it in at the lecture, or by 6:00 pm in the 8.07 homework boxes.

**READING ASSIGNMENT:** Chapter 5 of Griffiths: *Magnetostatics*.

**CREDIT:** This problem set has 115 points of credit, plus an option for 10 points of extra credit.

**PROBLEM 1: J.J. THOMPSON AND THE CHARGE TO MASS RATIO OF THE ELECTRON** (10 points)

Griffiths Problem 5.3 (p. 208).

**PROBLEM 2: EXAMPLES OF THE USE OF THE BIOT-SAVART LAW** (15 points)

Griffiths, Problem 5.8 (p. 219).

**PROBLEM 3: MAGNETIC FIELD ON THE AXIS OF A TIGHTLY WOUND SOLENOID** (15 points)

Consider a cylinder of radius of radius  $R$  centered on the  $z$ -axis, extending from  $z = z_1$  to  $z = z_2 > z_1$ . The cylinder is evenly wrapped with a wire which carries a current  $I$ , counterclockwise as seen from the positive  $z$  direction. The wire forms a helix, but we can approximate each turn as a circle in the  $x$ - $y$  plane, since the number  $n$  of turns per unit length is large compared to  $1/a$ .

- (a) (10 points) Calculate the magnetic field at the origin.
- (b) (5 points) What is  $\vec{B}$  at the origin in the limit of  $z_1 \rightarrow -\infty$  and  $z_2 \rightarrow \infty$ , corresponding to the infinite solenoid?

**PROBLEM 4: VECTOR POTENTIAL FOR A UNIFORM  $\vec{B}$  FIELD** (10 points)

Griffiths Problem 5.24 (p. 239).

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\* In the 2nd term of the 2nd line of Eq. (7.8), the argument of  $\varphi$  was mistyped as  $\vec{r}$  in the October 27 version of this problem set. The term should be  $\frac{4\pi}{3}\delta_{ij}\varphi(\vec{0})$ . Also, in the text between Eqs. (7.11) and (7.12), the equation  $d^3x = r^2 dr \sin\theta d\theta d\phi$  was mistyped in the earlier version.

**PROBLEM 5: A CHARGED PARTICLE IN THE FIELD OF A MAGNETIC MONOPOLE** (20 points)

Griffiths Problem 5.43 (p. 248), parts (a)–(c) (5 points each). Then follow this with

- (d) (5 points) Adopt a coordinate system so that the  $z$ -axis points in the direction of  $\vec{Q}$ , with the monopole at the origin. Then  $\vec{Q} = Q \hat{z}$ . Calculate  $\vec{Q} \cdot \hat{r}$ , and use the result to show that the motion is confined to a fixed value of the polar angle  $\theta$ , and hence to a cone. Express the value of  $\theta$  in terms of  $q_e$ ,  $q_m$ ,  $Q$ , and  $\mu_0$ .

**PROBLEM 6: THE MAGNETIC FIELD OF A SPINNING, UNIFORMLY CHARGED SPHERE** (25 points)

Griffiths Problem 5.58 (p. 253). In part (b), you are expected to use Eq. (5.89), from Problem 5.57, without proving it in any way. In part (c), where you are asked to find the approximate vector potential, you are expected to find it in the dipole approximation.

**PROBLEM 7: A DELTA FUNCTION IDENTITY AND ITS CONSEQUENCES** (20 points plus 10 points extra credit):

Essentially every E&M equation involving  $\delta$ -functions can be derived from the identity

$$\partial_i \left( \frac{\hat{r}_j}{r^2} \right) \equiv \partial_i \left( \frac{x_j}{r^3} \right) = \frac{\delta_{ij} - 3\hat{r}_i \hat{r}_j}{r^3} + \frac{4\pi}{3} \delta_{ij} \delta^3(\vec{r}) . \quad (7.1)$$

To make sure the definitions are clear, I remind you that  $\vec{r} \equiv x_i \hat{e}_i \equiv x \hat{e}_x + y \hat{e}_y + z \hat{e}_z$ ,  $r \equiv |\vec{r}|$ ,  $\hat{r} \equiv \vec{r}/r$ ,  $\hat{r}_i \equiv x_i/r$ , and  $\partial_i \equiv \partial/\partial x_i$ . (Despite the usefulness of this identity, I have not seen it in the literature. If anyone has, please tell me (Alan Guth) where.)

- (a) (5 points) Show that the famous identity,

$$\nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta^3(\vec{r}) , \quad (7.2)$$

follows immediately from Eq. (7.1). (*Hint*: recall that  $x_j/r^3 = -\partial_j(1/r)$ .)

- (b) (5 points) Using this identity, show that the potential for an electric dipole,

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r} \cdot \vec{p}}{r^2} , \quad (7.3)$$

can be differentiated straightforwardly to give the electric field of a dipole,

$$\vec{E}_{\text{dip}}(\vec{r}) = -\vec{\nabla} V_{\text{dip}} = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}}{r^3} - \frac{1}{3\epsilon_0} \vec{p} \delta^3(\vec{r}) . \quad (7.4)$$

- (c) (5 points) Again using this identity, show that the vector potential for a magnetic dipole,

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}, \quad (7.5)$$

can be used to find immediately that the magnetic field of a magnetic dipole is given by

$$\vec{B}_{\text{dip}}(\vec{r}) = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3} \vec{m} \delta^3(\vec{r}). \quad (7.6)$$

- (d) (5 points) Demonstrate that the identity holds by first showing that it is valid for  $r \neq 0$ , and then verifying the  $\delta$ -function by integrating over a small sphere of radius  $\epsilon$  about the origin. Use the identity of Problem 1.60 part (a), p. 56, to convert

$$\int_{r < \epsilon} \partial_i \left( \frac{\hat{r}_j}{r^2} \right) d^3x \quad (7.7)$$

into a surface integral on a sphere of radius  $\epsilon$ . (Here the integrand has an extra vector index  $j$ , but the identity in Problem 1.60(a) holds for any function  $T(\vec{r})$ ; if the function happens to be the  $j$ 'th component of a vector, the identity still applies.) Evaluate the surface integral to show that the  $\delta$ -function term in Eq. (7.1) is correct.

- (e) (10 points extra credit) Prove the identity (7.1), using the mathematical definitions of distribution theory. The calculation will not be much different from that of part (d), but the justification needs to be phrased more carefully. In particular, mathematicians would criticize the calculation in part (d) by saying that the integral in Eq. (7.7) was ill-defined from the beginning, since  $(\hat{r}_j/r^2)$  is not differentiable at  $r = 0$ .

Mathematicians give a rigorous meaning to the identity (7.1) by defining both sides as distributions, which means that they are defined by the result of multiplying them by an arbitrary test function  $\varphi(\vec{r})$  (which is assumed to be smooth and to fall off rapidly as  $r \rightarrow \infty$ ) and then integrating both sides over all space. (The distribution is thus a mapping from the test function  $\varphi(\vec{r})$  to a number, the value of the integral.) If the integrals obtained on each side of the equation are equal, then the identity (7.1) is valid — i.e., the two sides of Eq. (7.1) are said to be equal in the sense of distributions.\* For the right-hand side (RHS),

$$\begin{aligned} RHS &= \int \varphi(\vec{r}) \left[ \frac{\delta_{ij} - 3\hat{r}_i\hat{r}_j}{r^3} + \frac{4\pi}{3} \delta_{ij} \delta^3(\vec{r}) \right] d^3x \\ &= \int \varphi(\vec{r}) \frac{\delta_{ij} - 3\hat{r}_i\hat{r}_j}{r^3} d^3x + \frac{4\pi}{3} \delta_{ij} \varphi(\vec{0}). \end{aligned} \quad (7.8)$$

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\* Note, however, that “equal in the sense of distributions” might not correspond completely to your intuitive sense of “equal”. For example, a function of  $x$  that is equal to 1 at  $x = 0$  and zero everywhere else is equal in the sense of distributions to the function that is zero everywhere.

The integral above is singular, but may be defined precisely by

$$\int \varphi(\vec{r}) \frac{\delta_{ij} - 3\hat{r}_i\hat{r}_j}{r^3} d^3x \equiv \lim_{\epsilon \rightarrow 0} \int_{r>\epsilon} \varphi(\vec{r}) \frac{\delta_{ij} - 3\hat{r}_i\hat{r}_j}{r^3} d^3x . \quad (7.9)$$

In this form the integral is well-defined, since the measure  $d^3x = r^2 dr \sin\theta d\theta d\phi$  cancels two factors of  $r$  in the denominator, and the remaining factor of  $r$  is harmless because  $\delta_{ij} - 3\hat{r}_i\hat{r}_j$  vanishes when integrated over angles.

To evaluate the left-hand side (LHS), integrate by parts:

$$LHS = \int \varphi(\vec{r}) \partial_i \left( \frac{\hat{r}_j}{r^2} \right) d^3x \quad (7.10a)$$

$$= - \int \partial_i \varphi(\vec{r}) \left( \frac{\hat{r}_j}{r^2} \right) d^3x . \quad (7.10b)$$

Note that Eq. (7.10a) involves the same ill-defined derivative as Eq. (7.7), but the mathematicians give it unambiguous meaning by defining the derivative of a distribution by integration by parts: all the derivatives are applied to the test function  $\varphi(\vec{r})$ . Thus the LHS is defined by Eq. (7.10b).

To manipulate Eq. (7.10b), we first write it as

$$LHS = - \lim_{\epsilon \rightarrow 0} \int_{r>\epsilon} \partial_i \varphi(\vec{r}) \left( \frac{\hat{r}_j}{r^2} \right) d^3x . \quad (7.11)$$

The restriction of the integral to  $r > \epsilon$  makes no difference in the limit  $\epsilon \rightarrow 0$ , since the integrand behaves smoothly (recall that  $d^3x = r^2 dr \sin\theta d\theta d\phi$ ). By writing it this way, however, we can separate it into pieces which individually will not behave smoothly as  $r \rightarrow 0$ . Specifically, we can integrate by parts:

$$LHS = - \lim_{\epsilon \rightarrow 0} \int_{r>\epsilon} \left\{ \partial_i \left[ \varphi(\vec{r}) \left( \frac{\hat{r}_j}{r^2} \right) \right] - \varphi(\vec{r}) \partial_i \left( \frac{\hat{r}_j}{r^2} \right) \right\} d^3x . \quad (7.12)$$

Here's where you take over. Show that the second term in the integrand reproduces Eq. (7.9), and that the first term can be converted to a surface integral that evaluates to  $(4\pi/3)\delta_{ij}\varphi(\vec{r})$ .

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