MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — December 19, 2012
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FINAL EXAM

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THE FORMULA SHEETS ARE AT THE END OF THE EXAM.

Your Name ______________________ Recitation ______________

Problem	Maximum	Score
1	20	
2	20	
3	15	
4	20	
5	25	
TOTAL	100	

PROBLEM 1: ANGULAR MOMENTUM AND A ROTATING SHELL OF CHARGE *(20 points)*

This is an abbreviated version of Problem 3 of Problem Set 10.

A total charge Q is uniformly distributed over the surface of a sphere of radius R. The sphere rotates about the z axis with angular velocity ω. The magnetic field of this construction has been calculated, and is known to be

$$\vec{B} = \begin{cases} \vec{B}_{\rm in} = b_0 \omega\, \hat{z} & \text{if } r < R \\ \vec{B}_{\rm out} = \dfrac{b_0 \omega R^3}{2r^3}\Big(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\Big) & \text{if } r > R\ , \end{cases} \tag{1.1}$$

where

$$b_0 = \frac{\mu_0}{4\pi}\frac{2}{3}\frac{Q}{R}\ . \tag{1.2}$$

(a) *(7 points)* Consider the case where $\dot{\omega} \neq 0$. Calculate the Faraday induced electric field at the surface of the sphere as a function of θ. Calculate also the torque this field produces on the sphere.

(b) *(6 points)* Now assume that ω is constant. Calculate the energy stored in the magnetic field. Show that two-thirds is inside the sphere and one-third is outside the sphere. Write the total magnetic energy as $\frac{1}{2} I_{\rm mag}\omega^2$, and find an expression for $I_{\rm mag}$.

(c) *(7 points)* Calculate the angular momentum stored in the fields. Verify that the magnitude of the angular momentum coincides with $I_{\rm mag}\omega$.

PROBLEM 2: CONSERVATION OF MOMENTUM IN THE PRESENCE OF MAGNETIC MONOPOLES *(20 points)*

We have learned how to show, for the standard version of Maxwell's equations involving electric charge density ρ_e and current density $\vec{J}_e$, that momentum is conserved. In detail, we learned how to show that for any volume $\mathcal{V}$ bounded by a surface S,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P_{\mathrm{mech},i} + \frac{1}{c^2}\int_{\mathcal{V}} S_i\,\mathrm{d}^3x\right) = \oint_S T_{ij}\,\mathrm{d}a_j \ , \tag{2.1}$$

where $P_{\mathrm{mech},i}$ is the i'th component of the total mechanical momentum in $\mathcal{V}$, S_i is the i'th component of the Poynting vector, and T_{ij} is the Maxwell stress tensor. Explicit formulas for S_i and T_{ij} are given on p. 13 of the formula sheets. The goal of this problem is to extend the result to include the possibility that magnetic charges also exist, as described by the extended version of Maxwell's equations shown in the formula sheets on p. 11. To simplify the algebra, however, we will consider the case where **ONLY** magnetic charge density ρ_m and magnetic current density $\vec{J}_m$ are present, but $\rho_e = \vec{J}_e = 0$.

(a) *(5 points)* If the magnetic charges are discrete, then the rate of change of the total mechanical momentum in the region is just the sum of the forces acting on the magnetic monopoles:

$$\frac{\mathrm{d}P_{\mathrm{mech},i}}{\mathrm{d}t} = \sum_n F_i^{(n)} = \sum_n q_m^{(n)}\left(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}\right)_i \ , \tag{2.2}$$

where $F_i^{(n)}$ is the i'th component of the force on the n'th monopole, and $q_m^{(n)}$ is the magnetic charge of the n'th monopole. To demonstrate that Eq. (2.1) holds in the presence of monopoles, we need to write $\mathrm{d}P_{\mathrm{mech},i}/\mathrm{d}t$ in the form of a volume integral over a force density f_i,

$$\frac{\mathrm{d}P_{\mathrm{mech},i}}{\mathrm{d}t} = \int_{\mathcal{V}} f_i\,\mathrm{d}^3x \ , \tag{2.3}$$

which we do by thinking of each infinitesimal volume element as a magnetic monopole of magnetic charge $\rho_m\,\mathrm{d}^3x$. Write an expression for f_i in terms of ρ_m, $\vec{J}_m$, $\vec{E}$, and $\vec{B}$. Note that $\vec{v}$ should not appear in your answer.

(b) *(5 points)* Use Maxwell's equations to write the left-hand side of Eq. (2.1), with your result from part (a), entirely in terms of $\vec{E}$ and $\vec{B}$.

(c) *(10 points)* Complete the proof, showing that the left hand side of Eq. (2.1) can be written as a surface integral. Does the Maxwell stress tensor T_{ij} require any modification? (*Hint:* We recommend that you use index notation, but you may use vector notation if you prefer. The vector identities from the inside cover of the textbook have been added to the formula sheet.)

PROBLEM 3: THE ELECTRIC FIELD OF A CHARGED PARTICLE MOVING AT A CONSTANT VELOCITY *(15 points)*

Suppose that a particle of charge q is moving at speed v_0 along the x-axis, following the trajectory

$$\vec{r}(t) = v_0 t\,\hat{x} \; . \tag{3.1}$$

From differentiating the Liénard-Wiechert potentials, we have learned that the electric field of a point charge is given in general by

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{|\vec{r}-\vec{r}_p|}{\left(\vec{u}\cdot(\vec{r}-\vec{r}_p)\right)^3}\left[(c^2-v_p^2)\vec{u} + (\vec{r}-\vec{r}_p)\times(\vec{u}\times\vec{a}_p)\right] \; , \tag{3.2}$$

where the notation is defined in the formula sheets on p. 16. Suppose that an observer is located on the y axis at $(0, y_0, 0)$, and measures the electric field at time $t = 0$, exactly when the particle crosses the origin. Note that all measurements are made in the same coordinate system that was used in Eq. (3.1) to describe the trajectory.

(a) *(5 points)* Find the value of the retarded time t_r appropriate to this measurement, and also find the values of $\vec{r}-\vec{r}_p$ and $\vec{u}$. (Remember that vector quantities must be described as vectors, not numbers.)

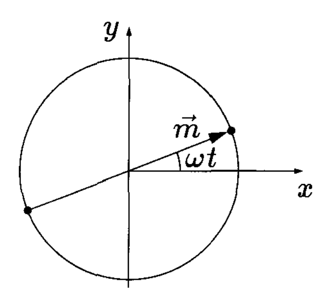
(b) *(5 points)* Use Eq. (3.2) to find the value of $\vec{E}$ that the observer will measure. (Some of you may know the answer to this question, but to get full credit you must show how to obtain it from Eq. (3.2).)

(c) *(5 points)* Find also the magnetic field $\vec{B}$ that the observer will measure at the same time.

PROBLEM 4: A ROTATING MAGNETIC DIPOLE *(20 points)*

A rotating **magnetic** dipole can be thought of as the superposition of two *oscillating* magnetic dipoles, one along the x axis, and the other along the y axis (see figure at the right), with the latter out of phase by 90°:

$$\vec{m} = m_0[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}] \; . \tag{4.1}$$



(a) *(10 points)* Suppose that an observer is located along the z axis, at $(0, 0, z_0)$. Find the electric and magnetic fields that she measures, as a function of time. (Remember that the electric and magnetic fields are vectors, not numbers.)

(b) *(5 points)* Find the Poynting vector measured by the observer in part (a), as a function of time, and also find the time-averaged value of the Poynting vector.

(c) *(5 points)* Now suppose that the observer was located along the x axis, at $(x_0, 0, 0)$. Find the electric and magnetic fields that she would measure at this point, as a function of time.

PROBLEM 5: SHORT QUESTIONS *(25 points)*

(a) *(5 points)* Suppose I invented a function $V_{\text{Alan}}(\vec{r},t)$, and defined the Alan gauge as the gauge for which $V(\vec{r},t) = V_{\text{Alan}}(\vec{r},t)$ for all $\vec{r}$ and all t. Is it always possible to write the electromagnetic potentials in Alan gauge? If so, give a recipe for finding the gauge function Λ that takes one to Alan gauge, where the gauge transformation is written as

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \ , \quad V' = V - \frac{\partial \Lambda}{\partial t} \ . \tag{5.1}$$

If not, give an argument why not.

(b) *(5 points)* Suppose I invented a vector function $\vec{A}_{\text{Alan}}(\vec{r},t)$, and defined the vector-Alan gauge as the gauge for which $\vec{A}(\vec{r},t) = \vec{A}_{\text{Alan}}(\vec{r},t)$ for all $\vec{r}$ and all t. Is it always possible to write the electromagnetic potentials in vector-Alan gauge? If so, give a recipe for finding the gauge function Λ that takes one to Alan gauge, and if not, give an argument why.

(c) *(7 points)* The potential energy function for an ideal (static) electric quadrupole is given by

$$V(\vec{r}) = \frac{1}{8\pi\epsilon_0} Q_{ij} \frac{\hat{r}_i \hat{r}_j}{r^3} \ , \tag{5.2}$$

where Q_{ij} is the quadrupole moment tensor, which is traceless and symmetric. Use Poisson's equation to find the charge density $\rho(\vec{r})$ for an ideal quadrupole. Your answer should be expressed in terms of Q_{ij}, and some kind of derivative of a delta function. (*Hint:* Start by using the identity

$$-\partial_i \partial_j \left(\frac{1}{r}\right) = \frac{\delta_{ij} - 3\hat{r}_i \hat{r}_j}{r^3} + \frac{4\pi}{3}\,\delta_{ij}\,\delta^3(\vec{r}) \tag{5.3}$$

from the formula sheet to express $\hat{r}_i \hat{r}_j / r^3$ in terms of other quantities.)

(d) *(8 points)* Accelerating charges normally radiate, but that is not always the case. Consider a sphere with a total charge Q uniformly spread on its surface, with a radius $R(t)$ which pulsates:

$$R(t) = R_0 + R_1 \sin \omega t \ , \tag{5.4}$$

where $0 < R_1 < R_0$, so $R(t)$ is always positive. Show that this system does not radiate, even though the charges are accelerating. (*Hint:* take advantage of the symmetry of the system to write down an exact solution to Maxwell's equations.)

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