MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 15, 2012

Prof. Alan Guth

QUIZ 2

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THE FORMULA SHEETS ARE AT THE END OF THE EXAM.

Your Name ______________________

Recitation ______________

Problem	**Maximum**	**Score**
1	25	
2	35	
3	20	
4	10	
5	10	
TOTAL	100	

* A few clarifications that were posted on the blackboard during the quiz are incorporated here into the text.

PROBLEM 1: THE MAGNETIC FIELD OF A SPINNING, UNIFORMLY CHARGED SPHERE *(25 points)*

This problem is based on Problem 1 of Problem Set 8.

A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) *(10 points)* What is the magnetic dipole moment of the sphere?

(b) *(5 points)* Using the dipole approximation, what is the vector potential $\vec{A}(\vec{r})$ at large distances? (Remember that $\vec{A}$ is a vector, so it is not enough to merely specify its magnitude.)

(c) *(10 points)* Find the exact vector potential *INSIDE* the sphere. You may, if you wish, make use of the result of Example 5.11 from Griffiths' book. There he considered a spherical shell, of radius R, carrying a uniform surface charge σ, spinning at angular velocity $\vec{\omega}$ directed along the z axis. He found the vector potential

$$\vec{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\phi}\ , & (\text{if } r \le R) \\ \dfrac{\mu_0 R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\,\hat{\phi}\ , & (\text{if } r \ge R)\ . \end{cases} \tag{1.1}$$

PROBLEM 2: SPHERE WITH VARIABLE DIELECTRIC CONSTANT *(35 points)*

A dielectric sphere of radius R has variable permittivity, so the permittivity throughout space is described by

$$\epsilon(r) = \begin{cases} \epsilon_0 (R/r)^2 & \text{if } r < R \\ \epsilon_0\ , & \text{if } r > R\ . \end{cases} \tag{2.1}$$

There are no free charges anywhere in this problem. The sphere is embedded in a constant external electric field $\vec{E} = E_0\hat{z}$, which means that $V(\vec{r}) \approx -E_0 r\cos\theta$ for $r \gg R$.

(a) *(9 points)* Show that $V(\vec{r})$ obeys the differential equation

$$\nabla^2 V + \frac{\mathrm{d}\ln\epsilon}{\mathrm{d}r}\frac{\partial V}{\partial r} = 0 \tag{2.2}$$

for all $\vec{r}$, as a consequence of the laws of electrostatics.

(b) *(4 points)* Explain why the solution can be written as

$$V(r,\theta) = \sum_{\ell=0}^{\infty} V_\ell(r)\{\,\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}\,\hat{r}_{i_1}\ldots\hat{r}_{i_\ell}\ , \tag{2.3a}$$

or equivalently (your choice)

$$V(r,\theta) = \sum_{\ell=0}^{\infty} V_\ell(r) P_\ell(\cos\theta) \ , \tag{2.3b}$$

where $\{\ldots\}$ denotes the traceless symmetric part of $\ldots$, and $P_\ell(\cos\theta)$ is the Legendre polynomial. (Your answer here should depend only on general mathematical principles, and should not rely on the explicit solution that you will find in parts (c) and (d).)

(c) *(9 points)* Derive the ordinary differential equation obeyed by $V_\ell(r)$ (separately for $r < R$ and $r > R$) and give its two independent solutions in each region. *Hint:* they are powers of r. You may want to know that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}P_\ell(\cos\theta)}{\mathrm{d}\theta}\right) = -\ell(\ell+1)\sin\theta P_\ell(\cos\theta) \ . \tag{2.4}$$

The relevant formulas for the traceless symmetric tensor formalism are in the formula sheets.

(d) *(9 points)* Using appropriate boundary conditions on $V(r,\theta)$ at $r = 0$, $r = R$, and $r \to \infty$, determine $V(r,\theta)$ for $r < R$ and $r > R$.

(e) *(4 points)* What is the net dipole moment of the polarized sphere?

PROBLEM 3: PAIR OF MAGNETIC DIPOLES *(20 points)*

Suppose there are two magnetic dipoles. One has dipole moment $\vec{m}_1 = m_0\hat{z}$ and is located at $\vec{r}_1 = +\frac{1}{2}a\,\hat{z}$; the other has dipole moment $\vec{m}_2 = -m_0\hat{z}$, and is located at $\vec{r}_2 = -\frac{1}{2}a\,\hat{z}$.

(a) *(10 points)* For a point on the z axis at large z, find the leading (in powers of $1/z$) behavior for the vector potential $\vec{A}(0,0,z)$ and the magnetic field $\vec{B}(0,0,z)$.

(b) *(3 points)* In the language of monopole ($\ell = 0$), dipole ($\ell = 1$), quadrupole ($\ell = 2$), octupole ($\ell = 3$), etc., what type of field is produced at large distances by this current configuration? In future parts, the answer to this question will be called a whatapole.

(c) *(3 points)* We can construct an ideal whatapole — a whatapole of zero size — by taking the limit as $a \to 0$, keeping $m_0 a^n$ fixed, for some power n. What is the correct value of n?

(d) *(4 points)* Given the formula for the current density of a dipole,

$$\vec{J}_{\text{dip}}(\vec{r}) = -\vec{m} \times \vec{\nabla}_{\vec{r}}\,\delta^3(\vec{r} - \vec{r}_d) \ , \tag{3.1}$$

where $\vec{r}_d$ is the position of the dipole, find an expression for the current density of the whatapole constructed in part (c). Like the above equation, it should be expressed in terms of δ-functions and/or derivatives of δ-functions, and maybe even higher derivatives of δ-functions.

PROBLEM 4: UNIFORMLY MAGNETIZED INFINITE CYLINDER *(10 points)*

Consider a uniformly magnetized infinite circular cylinder, of radius R, with its axis coinciding with the z axis. The magnetization inside the cylinder is $\vec{M} = M_0 \hat{z}$.

(a) *(5 points)* Find $\vec{H}(\vec{r})$ everywhere in space.

(b) *(5 points)* Find $\vec{B}(\vec{r})$ everywhere in space.

PROBLEM 5: ELECTRIC AND MAGNETIC UNIFORMLY POLARIZED SPHERES *(10 points)*

Compare the electric field of a uniformly polarized sphere with the magnetic field of a uniformly magnetized sphere; in each case the dipole moment per unit volume points along $\hat{z}$. Multiple choice: which of the following is true?

(a) The $\vec{E}$ and $\vec{B}$ field lines point in the same direction both inside and outside the spheres.

(b) The $\vec{E}$ and $\vec{B}$ field lines point in the same direction inside the spheres but in opposite directions outside.

(c) The $\vec{E}$ and $\vec{B}$ field lines point in opposite directions inside the spheres but in the same direction outside.

(d) The $\vec{E}$ and $\vec{B}$ field lines point in opposite directions both inside and outside the spheres.

No justification needed. (But if you give a justification, there is a chance that you might get partial credit for an incorrect answer.)

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II November 13, 2012
Prof. Alan Guth

FORMULA SHEET FOR QUIZ 2, V. 2

Exam Date: November 15, 2012

*** Some sections below are marked with asterisks, as this section is. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

Index Notation:

$$\vec{A}\cdot\vec{B} = A_i B_i\,, \qquad \vec{A}\times\vec{B}_i = \epsilon_{ijk} A_j B_k\,, \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n}$$

Rotation of a Vector:

$$A'_i = R_{ij} A_j\,, \qquad \text{Orthogonality: } R_{ij} R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

Rotation about z-axis by ϕ: $$R_z(\phi)_{ij} = \begin{array}{c c} & \begin{array}{ccc} \scriptstyle j=1 & \scriptstyle j=2 & \scriptstyle j=3 \end{array} \\ \begin{array}{c} \scriptstyle i=1 \\ \scriptstyle i=2 \\ \scriptstyle i=3 \end{array} & \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rotation about axis $\hat{n}$ by ϕ:***

$$R(\hat{n},\phi)_{ij} = \delta_{ij}\cos\phi + \hat{n}_i\hat{n}_j(1-\cos\phi) - \epsilon_{ijk}\hat{n}_k \sin\phi\,.$$

Vector Calculus:

Gradient: $$(\vec{\nabla}\varphi)_i = \partial_i\varphi\,, \qquad \partial_i \equiv \frac{\partial}{\partial x_i}$$

Divergence: $$\vec{\nabla}\cdot\vec{A} \equiv \partial_i A_i$$

Curl: $$(\vec{\nabla}\times\vec{A})_i = \epsilon_{ijk}\partial_j A_k$$

Laplacian: $$\nabla^2\varphi = \vec{\nabla}\cdot(\vec{\nabla}\varphi) = \frac{\partial^2\varphi}{\partial x_i \partial x_i}$$

Fundamental Theorems of Vector Calculus:

Gradient: $$\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\varphi\cdot \mathrm{d}\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$$

Divergence: $$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{A}\,\mathrm{d}^3x = \oint_S \vec{A}\cdot\mathrm{d}\vec{a}$$

where S is the boundary of $\mathcal{V}$

Curl: $$\int_S (\vec{\nabla}\times\vec{A})\cdot\mathrm{d}\vec{a} = \oint_P \vec{A}\cdot\mathrm{d}\vec{\ell}$$

where P is the boundary of S

Delta Functions:

$$\int \varphi(x)\delta(x-x')\,\mathrm{d}x = \varphi(x')\,, \qquad \int \varphi(\vec{r})\delta^3(\vec{r}-\vec{r}\,')\,\mathrm{d}^3x = \varphi(\vec{r}\,')$$

$$\int \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\,\mathrm{d}x = -\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,, \quad g(x_i)=0$$

$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right) = -\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\delta^3(\vec{r}-\vec{r}\,')$$

$$\partial_i\left(\frac{\hat{r}_j}{r^2}\right) \equiv \partial_i\left(\frac{x_j}{r^3}\right) = -\partial_i\partial_j\left(\frac{1}{r}\right) = \frac{\delta_{ij}-3\hat{r}_i\hat{r}_j}{r^3} + \frac{4\pi}{3}\,\delta_{ij}\,\delta^3(\vec{r})$$

$$\vec{\nabla}\cdot\frac{3(\vec{d}\cdot\hat{r})\hat{r}-\vec{d}}{r^3} = -\frac{8\pi}{3}(\vec{d}\cdot\vec{\nabla})\delta^3(\vec{r})$$

$$\vec{\nabla}\times\frac{3(\vec{d}\cdot\hat{r})\hat{r}-\vec{d}}{r^3} = -\frac{4\pi}{3}\vec{d}\times\vec{\nabla}\delta^3(\vec{r})$$

Electrostatics:

$\vec{F} = q\vec{E}$, where

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\vec{r}-\vec{r}\,')\,q_i}{|\vec{r}-\vec{r}\,'|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\rho(\vec{r}\,')\,\mathrm{d}^3x'$$

ϵ_0 =permittivity of free space $= 8.854\times10^{-12}\ \mathrm{C^2/(N{\cdot}m^2)}$

$$\frac{1}{4\pi\epsilon_0} = 8.988\times10^{9}\ \mathrm{N{\cdot}m^2/C^2}$$

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot\mathrm{d}\vec{\ell}\,' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\mathrm{d}^3x'$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}\,, \qquad \vec{\nabla}\times\vec{E} = 0\,, \qquad \vec{E} = -\vec{\nabla}V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}\ \text{(Poisson's Eq.)}\,, \qquad \rho = 0 \implies \nabla^2 V = 0\ \text{(Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

Energy:

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij\\ i\neq j}} \frac{q_i q_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$W = \frac{1}{2}\int \mathrm{d}^3x\,\rho(\vec{r})V(\vec{r}) = \frac{1}{2}\epsilon_0\int \left|\vec{E}\right|^2\mathrm{d}^3x$$

Conductors:

Just outside, $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{n}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j \; , \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij} V_j \; , \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\dfrac{a}{R}Q$, $r = \dfrac{a^2}{R}$

Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \left\{ \begin{array}{c} \cos \alpha x \\ \sin \alpha x \end{array} \right\} \left\{ \begin{array}{c} \cos \beta y \\ \sin \beta y \end{array} \right\} \left\{ \begin{array}{c} \cosh \gamma z \\ \sinh \gamma z \end{array} \right\} \qquad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

Separation of Variables for Laplace's Equation in Spherical Coordinates:

Traceless Symmetric Tensor expansion:

$$\nabla^2 \varphi(r, \theta, \phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\nabla_\theta^2 \varphi = 0 \; ,$$

where the angular part is given by

$$\nabla_\theta^2 \varphi \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \varphi}{\partial \phi^2}$$

$$\nabla_\theta^2 \, C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} = -\ell(\ell+1) C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \; ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a symmetric traceless tensor and

$\hat{n} = \sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3$.

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty}\left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}}\right)\hat{r}_{i_1}\hat{r}_{i_2}\ldots\hat{r}_{i_\ell} \; , \quad \text{where } \vec{r} = r\hat{r}$$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \{ \, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \, \} \, \hat{r}_{i_1} \ldots \hat{r}_{i_\ell}$$

where $\{ \ldots \}$ denotes the traceless symmetric part of $\ldots$.

Special cases:

$$\{ \, 1 \, \} = 1$$

$$\{ \, \hat{z}_i \, \} = \hat{z}_i$$

$$\{ \, \hat{z}_i \hat{z}_j \, \} = \hat{z}_i \hat{z}_j - \tfrac{1}{3} \delta_{ij}$$

$$\{ \, \hat{z}_i \hat{z}_j \hat{z}_k \, \} = \hat{z}_i \hat{z}_j \hat{z}_k - \tfrac{1}{5} \big(\hat{z}_i \delta_{jk} + \hat{z}_j \delta_{ik} + \hat{z}_k \delta_{ij} \big)$$

$$\begin{aligned} \{ \, \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \, \} = {} & \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m - \tfrac{1}{7} \big(\hat{z}_i \hat{z}_j \delta_{km} + \hat{z}_i \hat{z}_k \delta_{mj} + \hat{z}_i \hat{z}_m \delta_{jk} + \hat{z}_j \hat{z}_k \delta_{im} \\ & + \hat{z}_j \hat{z}_m \delta_{ik} + \hat{z}_k \hat{z}_m \delta_{ij} \big) + \tfrac{1}{35} \big(\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk} \big) \end{aligned}$$

Legendre Polynomial / Spherical Harmonic expansion:

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} \, r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

Orthonormality: $\displaystyle \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, Y^*_{\ell' m'}(\theta,\phi) \, Y_{\ell m}(\theta,\phi) = \delta_{\ell' \ell} \delta_{m' m}$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Electric Multipole Expansion:

First several terms:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\, \frac{Q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{1}{2} \frac{\hat{r}_i \hat{r}_j}{r^3} \, Q_{ij} + \cdots \right] \quad \text{, where}$$

$$Q = \int \mathrm{d}^3 x \, \rho(\vec{r}) \, , \quad p_i = \int d^3 x \, \rho(\vec{r}) \, x_i \quad Q_{ij} = \int d^3 x \, \rho(\vec{r}) (3 x_i x_j - \delta_{ij} |\vec{r}|^2) \, ,$$

$$\vec{E}_{\text{dip}}(\vec{r}) = -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{\vec{p} \cdot \hat{r}}{r^2} \right) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{r})$$

$$\vec{\nabla} \times \vec{E}_{\text{dip}}(\vec{r}) = 0 \, , \qquad \vec{\nabla} \cdot \vec{E}_{\text{dip}}(\vec{r}) = \frac{1}{\epsilon_0} \rho_{\text{dip}}(\vec{r}) = -\frac{1}{\epsilon_0} \vec{p} \cdot \vec{\nabla} \delta^3(\vec{r})$$

Traceless Symmetric Tensor version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1 \ldots i_\ell} \hat{r}_{i_1} \ldots \hat{r}_{i_\ell} \ ,$$

where

$$C^{(\ell)}_{i_1 \ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}) \{ x_{i_1} \ldots x_{i_\ell} \} \, \mathrm{d}^3 x \qquad (\vec{r} \equiv r\hat{r} \equiv x_i \hat{e}_i)$$

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^{\ell}}{r^{\ell+1}} \{ \hat{r}_{i_1} \ldots \hat{r}_{i_\ell} \} \hat{r}'_{i_1} \ldots \hat{r}'_{i_\ell} \ , \qquad \text{for } r' < r$$

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\ldots 1 = \frac{(2\ell)!}{2^\ell \ell!} \ , \text{ with } \ (-1)!! \equiv 1 \ .$$

Reminder: $\{\ldots\}$ denotes the traceless symmetric part of $\ldots$.

Griffiths version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^{\ell} \rho(\vec{r}\,') P_\ell(\cos\theta') \, \mathrm{d}^3 x$$

where θ' = angle between $\vec{r}$ and $\vec{r}\,'$.

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta') \, , \qquad \frac{1}{\sqrt{1 - 2\lambda x + \lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^\ell (x^2 - 1)^\ell \, , \qquad \text{(Rodrigues' formula)}$$

$$P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^{1} \mathrm{d}x \, P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1} \delta_{\ell'\ell}$$

Spherical Harmonic version:***

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta, \phi)$$

where $q_{\ell m} = \int Y^*_{\ell m} r'^{\ell} \rho(\vec{r}\,') \, \mathrm{d}^3 x'$

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi) \ , \qquad \text{for } r' < r$$

Electric Fields in Matter:

Electric Dipoles:

$$\vec{p} = \int d^3x \, \rho(\vec{r}) \, \vec{r}$$

$\rho_{\text{dip}}(\vec{r}) = -\vec{p} \cdot \vec{\nabla}_{\vec{r}} \, \delta^3(\vec{r} - \vec{r}_d)$, where $\vec{r}_d$ = position of dipole

$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} = \vec{\nabla}(\vec{p} \cdot \vec{E})$ (force on a dipole)

$\vec{\tau} = \vec{p} \times \vec{E}$ (torque on a dipole)

$U = -\vec{p} \cdot \vec{E}$

Electrically Polarizable Materials:

$\vec{P}(\vec{r})$ = polarization = electric dipole moment per unit volume

$\rho_{\text{bound}} = -\nabla \cdot \vec{P}$, $\sigma_{\text{bound}} = \vec{P} \cdot \hat{n}$

$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$, $\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$, $\vec{\nabla} \times \vec{E} = 0$ (for statics)

Boundary conditions:

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_{\text{free}}$$

$$\vec{E}^{\parallel}_{\text{above}} - \vec{E}^{\parallel}_{\text{below}} = 0 \qquad \vec{D}^{\parallel}_{\text{above}} - \vec{D}^{\parallel}_{\text{below}} = \vec{P}^{\parallel}_{\text{above}} - \vec{P}^{\parallel}_{\text{below}}$$

Linear Dielectrics:

$\vec{P} = \epsilon_0 \chi_e \vec{E}$, χ_e = electric susceptibility

$\epsilon \equiv \epsilon_0(1 + \chi_e)$ = permittivity, $\vec{D} = \epsilon \vec{E}$

$\epsilon_r = \dfrac{\epsilon}{\epsilon_0} = 1 + \chi_e$ = relative permittivity, or dielectric constant

Clausius-Mossotti equation: $\chi_e = \dfrac{N\alpha/\epsilon_0}{1 - \frac{N\alpha}{3\epsilon_0}}$, where N = number density of atoms or (nonpolar) molecules, α = atomic/molecular polarizability ($\vec{P} = \alpha \vec{E}$)

Energy: $W = \dfrac{1}{2} \displaystyle\int \vec{D} \cdot \vec{E} \, \mathrm{d}^3x$ (linear materials only)

Force on a dielectric: $\vec{F} = -\vec{\nabla} W$ (Even if one or more potential differences are held fixed, the force can be found by computing the gradient with the total charge on each conductor fixed.)

Magnetostatics:

Magnetic Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \, , \qquad \text{where } \vec{p} = \gamma m_0 \vec{v} \, , \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec{F} = \int I \mathrm{d}\vec{\ell} \times \vec{B} = \int \vec{J} \times \vec{B}\, \mathrm{d}^3x$$

Current Density:

Current through a surface S: $I_S = \int_S \vec{J} \cdot \mathrm{d}\vec{a}$

Charge conservation: $\dfrac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$

Moving density of charge: $\vec{J} = \rho \vec{v}$

Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\ell}' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\, \mathrm{d}a'$$

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\, \mathrm{d}^3x$$

where μ_0 = permeability of free space $\equiv 4\pi \times 10^{-7}$ N/A^2

Examples:

Infinitely long straight wire: $\vec{B} = \dfrac{\mu_0 I}{2\pi r}\, \hat{\phi}$

Infintely long tightly wound solenoid: $\vec{B} = \mu_0 n I_0\, \hat{z}$, where n = turns per unit length

Loop of current on axis: $\vec{B}(0,0,z) = \dfrac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}}\, \hat{z}$

Infinite current sheet: $\vec{B}(\vec{r}) = \dfrac{1}{2}\mu_0 \vec{K} \times \hat{n}$, $\hat{n}$ = unit normal toward $\vec{r}$

Vector Potential:

$$\vec{A}(\vec{r})_{\text{coul}} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, \mathrm{d}^3x' \,, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \,, \qquad \vec{\nabla} \cdot \vec{A}_{\text{coul}} = 0$$

$\vec{\nabla} \cdot \vec{B} = 0$ (Subject to modification if magnetic monopoles are discovered)

Gauge Transformations: $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Lambda(\vec{r})$ for any $\Lambda(\vec{r})$. $\vec{B} = \vec{\nabla} \times \vec{A}$ is unchanged.

Ampère's Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \,, \quad \text{or equivalently} \quad \int_P \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enc}}$$

Magnetic Multipole Expansion:

Traceless Symmetric Tensor version:

$$A_j(\vec r) = \frac{\mu_0}{4\pi}\sum_{\ell=0}^{\infty}\mathcal{M}^{(\ell)}_{j;i_1 i_2\ldots i_\ell}\frac{\{\,\hat r_{i_1}\ldots\hat r_{i_\ell}\,\}}{r^{\ell+1}}$$

$$\text{where}\quad \mathcal{M}^{(\ell)}_{j;i_1 i_2\ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!}\int \mathrm{d}^3x J_j(\vec r)\{\,x_{i_1}\ldots x_{i_\ell}\,\}$$

Current conservation restriction: $\displaystyle\int \mathrm{d}^3x \operatorname*{Sym}_{i_1\ldots i_\ell}(x_{i_1}\ldots x_{i_{\ell-1}}J_{i_\ell}) = 0$

where $\operatorname*{Sym}_{i_1\ldots i_\ell}$ means to symmetrize — i.e. average over all orderings — in the indices $i_1\ldots i_\ell$

Special cases:

$$\ell = 1: \quad \int \mathrm{d}^3x\, J_i = 0$$

$$\ell = 2: \quad \int \mathrm{d}^3x\,(J_i x_j + J_j x_i) = 0$$

Leading term (dipole): $\displaystyle\vec A(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec m\times\hat r}{r^2}$,

where

$$m_i = -\frac{1}{2}\epsilon_{ijk}\mathcal{M}^{(1)}_{j;k}$$

$$\vec m = \frac{1}{2}I\int_P \vec r\times \mathrm{d}\vec\ell = \frac{1}{2}\int \mathrm{d}^3x\,\vec r\times\vec J = I\vec a\ ,$$

where $\displaystyle\vec a = \int_S \mathrm{d}\vec a$ for any surface S spanning P

$$\vec B_{\rm dip}(\vec r) = \frac{\mu_0}{4\pi}\vec\nabla\times\frac{\vec m\times\hat r}{r^2} = \frac{\mu_0}{4\pi}\frac{3(\vec m\cdot\hat r)\hat r - \vec m}{r^3} + \frac{2\mu_0}{3}\,\vec m\,\delta^3(\vec r)$$

$$\vec\nabla\cdot\vec B_{\rm dip}(\vec r) = 0\ , \qquad \vec\nabla\times\vec B_{\rm dip}(\vec r) = \mu_0\vec J_{\rm dip}(\vec r) = -\mu_0\vec m\times\vec\nabla\delta^3(\vec r)$$

Griffiths version:

$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\oint (r')^\ell P_\ell(\cos\theta')\mathrm{d}\vec\ell'$$

Magnetic Fields in Matter:

Magnetic Dipoles:

$$\vec m = \frac{1}{2}I\int_P \vec r\times \mathrm{d}\vec\ell = \frac{1}{2}\int \mathrm{d}^3x\,\vec r\times\vec J = I\vec a$$

$\vec{J}_{\rm dip}(\vec{r}) = -\vec{m} \times \vec{\nabla}_{\vec{r}}\delta^3(\vec{r} - \vec{r}_d)$, where $\vec{r}_d$ = position of dipole

$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$ (force on a dipole)

$\vec{\tau} = \vec{m} \times \vec{B}$ (torque on a dipole)

$U = -\vec{m} \cdot \vec{B}$

Magnetically Polarizable Materials:

$\vec{M}(\vec{r})$ = magnetization = magnetic dipole moment per unit volume

$\vec{J}_{\rm bound} = \vec{\nabla} \times \vec{M}$, $\vec{K}_{\rm bound} = \vec{M} \times \hat{n}$

$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$, $\vec{\nabla} \times \vec{H} = \vec{J}_{\rm free}$, $\vec{\nabla} \cdot \vec{B} = 0$

Boundary conditions:

$B^\perp_{\rm above} - B^\perp_{\rm below} = 0$ $H^\perp_{\rm above} - H^\perp_{\rm below} = -(M^\perp_{\rm above} - M^\perp_{\rm below})$

$\vec{B}^\parallel_{\rm above} - \vec{B}^\parallel_{\rm below} = \mu_0(\vec{K} \times \hat{n})$ $\vec{H}^\parallel_{\rm above} - \vec{H}^\parallel_{\rm below} = \vec{K}_{\rm free} \times \hat{n}$

Linear Magnetic Materials:

$\vec{M} = \chi_m \vec{H}$, χ_m = magnetic susceptibility

$\mu = \mu_0(1 + \chi_m)$ = permeability, $\vec{B} = \mu\vec{H}$

Magnetic Monopoles:

$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{r}$; Force on a static monopole: $\vec{F} = q_m\vec{B}$

Angular momentum of monopole/charge system: $\vec{L} = \frac{\mu_0 q_e q_m}{4\pi}\hat{r}$, where $\hat{r}$ points from q_e to q_m

Dirac quantization condition: $\frac{\mu_0 q_e q_m}{4\pi} = \frac{1}{2}\hbar \times$ integer

Connection Between Traceless Symmetric Tensors and Legendre Polynomials or Spherical Harmonics:

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \,\}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$$

For $m \geq 0$,

$$Y_{\ell m}(\theta,\phi) = C^{(\ell,m)}_{i_1 \ldots i_\ell}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} ,$$

where $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} = d_{\ell m}\,\{\, \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \,\}$,

with $d_{\ell m} = \frac{(-1)^m(2\ell)!}{2^\ell \ell!}\sqrt{\frac{2^m\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}}$,

and $\hat{u}^+ = \frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$

Form $m < 0$, $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$

More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)e^{im\phi}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$$

Legendre Polynomials:

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \tfrac{1}{2}(3x^2-1)\\
P_3(x) &= \tfrac{1}{2}(5x^3-3x)\\
P_4(x) &= \tfrac{1}{8}(35x^4-30x^2+3)
\end{aligned}$$

SPHERICAL HARMONICS $Y_{lm}(\theta,\varphi)$

l = 0

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

l = 1

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

l = 2

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\varphi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\right)$$

l = 3

$$Y_{33} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta e^{3i\varphi}$$

$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta e^{2i\varphi}$$

$$Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta\,(5\cos^2\theta-1)e^{i\varphi}$$

$$Y_{30} = \sqrt{\frac{7}{4\pi}}\left(\tfrac{5}{2}\cos^3\theta-\tfrac{3}{2}\cos\theta\right)$$

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