Physics 8.07: Electromagnetism II   November 21, 2012
Prof. Alan Guth

QUIZ 2 SOLUTIONS
QUIZ DATE: NOVEMBER 15, 2012

PROBLEM 1: THE MAGNETIC FIELD OF A SPINNING, UNIFORMLY CHARGED SPHERE *(25 points)*

This problem is based on Problem 1 of Problem Set 8.

A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) *(10 points)* What is the magnetic dipole moment of the sphere?

(b) *(5 points)* Using the dipole approximation, what is the vector potential $\vec{A}(\vec{r})$ at large distances? (Remember that $\vec{A}$ is a vector, so it is not enough to merely specify its magnitude.)

(c) *(10 points)* Find the exact vector potential *INSIDE* the sphere. You may, if you wish, make use of the result of Example 5.11 from Griffiths' book. There he considered a spherical shell, of radius R, carrying a uniform surface charge σ, spinning at angular velocity $\vec{\omega}$ directed along the z axis. He found the vector potential

$$\vec{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} r \sin\theta \, \hat{\phi} , & (\text{if } r \leq R) \\[2mm] \dfrac{\mu_0 R^4 \omega \sigma}{3} \dfrac{\sin\theta}{r^2} \, \hat{\phi} , & (\text{if } r \geq R) \ . \end{cases} \tag{1.1}$$

PROBLEM 1 SOLUTION:

(a) A uniformly charged solid sphere of radius R carries a total charge Q, hence it has charge density $\rho = Q/(\frac{4}{3}\pi R^3)$. To find the magnetic moment of sphere we can divide the sphere into infinitesimal charges. Using spherical polar coordinates, we can take $dq = \rho \, d\tau = \rho \, r^2 \, dr \, \sin\theta \, d\theta \, d\phi$, with the contribution to the dipole moment given by $d\vec{m} = \frac{1}{2}\vec{r} \times \vec{J} d\tau$. One method would be to write down the volume integral directly, using $\vec{J} = \rho\vec{v} = \rho\vec{\omega} \times \vec{r}$. We can, however, integrate over ϕ before we start, so we are breaking the sphere into rings, where a given ring is indicated by its coordinates r and θ, and its size dr and $d\theta$. The volume of each ring is $d\tau = 2\pi r^2 \, dr \, \sin\theta \, d\theta$. The current dI in the ring is given by dq/T, where $T = 2\pi/\omega$ is the period, so

$$dI = \frac{dq}{T} = \frac{\omega \rho d\tau}{2\pi} = \omega \rho r^2 \, dr \, \sin\theta \, d\theta \ . \tag{1.2}$$

The magnetic dipole moment of each ring is then given by

$$
\mathrm{d}\vec{m}_{\mathrm{ring}} = \frac{1}{2} \int_{\mathrm{ring}} \vec{r} \times \vec{J}\, \mathrm{d}\tau = \frac{1}{2}\mathrm{d}I \int_{\mathrm{ring}} \vec{r} \times d\vec{\ell} = \mathrm{d}I(\pi r^2 \sin^2 \theta)\,\hat{z}\ . \qquad (1.3)
$$

The total magnetic dipole moment is then

$$
\vec{m} = \int \omega \rho r^2 \sin\theta\,(\pi r^2 \sin^2 \theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\hat{z}
$$

$$
= \pi\omega\rho \int_0^R r^4\,\mathrm{d}r \int_0^\pi (1 - \cos^2 \theta)\sin\theta\,\mathrm{d}\theta\,\hat{z}
$$

$$
= \pi\omega \frac{Q}{\frac{4}{3}\pi R^3} \frac{R^5}{5}\frac{4}{3} = \boxed{\ \frac{1}{5}Q\omega R^2\,\hat{z}\ .\ } \qquad (1.4)
$$

(b) The vector potential in dipole approximation is,

$$
\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{|\vec{m}|\sin\theta}{r^2}\,\hat{\phi} = \boxed{\ \frac{\mu_0}{4\pi}\frac{Q\omega R^2}{5}\frac{\sin\theta}{r^2}\,\hat{\phi}\ .\ } \qquad (1.5)
$$

(c) To calculate the exact vector potential inside the sphere, we split the sphere into shells. Let r' be the integration variable and the radius of a shell, moreover let dr' denote the thickness of the shell. Then we can use the results of Example 5.11 (pp. 236-37) in Griffiths, if we replace σ by its value for this case. The value of σ is found equating charges

$$
\sigma(4\pi r'^2) = \frac{Q}{\frac{4}{3}\pi R^3}(4\pi r'^2)dr' \qquad (1.6)
$$

and therefore we must replace

$$
\sigma \to \frac{Q}{\frac{4}{3}\pi R^3}dr'\ .
$$

Making this replacement in Griffiths' Eq. (5.67), quoted above as Eq. (1.1), we now have

$$
\mathrm{d}A_\phi(r,\theta,\phi) = \frac{Q}{\frac{4}{3}\pi R^3}dr'\,\frac{\mu_0\omega}{3}\sin\theta \begin{cases} r'r & \text{if } r < r' \\[2mm] \dfrac{r'^4}{r^2} & \text{if } r > r' \end{cases} \qquad (1.7)
$$

Note that the R of Griffiths has been replaced by r', which is the radius of the integration shell. Now we can calculate the vector potential inside the sphere at

some radius $r < R$. The integration will require two pieces, a piece where $0 < r' < r$ and the other where $r < r' < R$, thus using the two options in Eq. (1.7):

$$A_\phi(r, \theta, \phi) = \frac{\mu_0}{4\pi} \frac{Q\omega}{R^3} \sin\theta \left[\int_0^r dr' \frac{r'^4}{r^2} + \int_r^R dr' rr' \right] . \tag{1.8}$$

Doing the integrals one finds

$$A_\phi(r, \theta, \phi) = \frac{\mu_0}{4\pi} \frac{Q\omega}{R^3} \sin\theta \left[-\frac{3r^3}{10} + \frac{rR^2}{2} \right] . \tag{1.9}$$

PROBLEM 2: SPHERE WITH VARIABLE DIELECTRIC CONSTANT *(35 points)*

A dielectric sphere of radius R has variable permittivity, so the permittivity throughout space is described by

$$\epsilon(r) = \begin{cases} \epsilon_0 (R/r)^2 & \text{if } r < R \\ \epsilon_0 , & \text{if } r > R . \end{cases} \tag{2.1}$$

There are no free charges anywhere in this problem. The sphere is embedded in a constant external electric field $\vec{E} = E_0 \hat{z}$, which means that $V(\vec{r}) \equiv -E_0 r \cos\theta$ for $r \gg R$.

(a) *(9 points)* Show that $V(\vec{r})$ obeys the differential equation

$$\nabla^2 V + \frac{\mathrm{d}\ln\epsilon}{\mathrm{d}r} \frac{\partial V}{\partial r} = 0 . \tag{2.2}$$

(b) *(4 points)* Explain why the solution can be written as

$$V(r, \theta) = \sum_{\ell=0}^{\infty} V_\ell(r) \{ \hat{z}_{i_1} \dots \hat{z}_{i_\ell} \} \hat{r}_{i_1} \dots \hat{r}_{i_\ell} , \tag{2.3a}$$

or equivalently (your choice)

$$V(r, \theta) = \sum_{\ell=0}^{\infty} V_\ell(r) P_\ell(\cos\theta) , \tag{2.3b}$$

where $\{ \dots \}$ denotes the traceless symmetric part of $\dots$, and $P_\ell(\cos\theta)$ is the Legendre polynomial. (Your answer here should depend only on general mathematical principles, and should not rely on the explicit solution that you will find in parts (c) and (d).)

(c) *(9 points)* Derive the ordinary differential equation obeyed by $V_\ell(r)$ (separately for $r < R$ and $r > R$) and give its two independent solutions in each region. *Hint:* they are powers of r. You may want to know that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P_\ell(\cos\theta)}{\mathrm{d}\theta}\right) = -\ell(\ell+1)\sin\theta P_\ell(\cos\theta) . \tag{2.4}$$

The relevant formulas for the traceless symmetric tensor formalism are in the formula sheets.

(d) *(9 points)* Using appropriate boundary conditions on $V(r,\theta)$ at $r = 0$, $r = R$, and $r \to \infty$, determine $V(r,\theta)$ for $r < R$ and $r > R$.

(e) *(4 points)* What is the net dipole moment of the polarized sphere?

PROBLEM 2 SOLUTION:

(a) Since we don't have free charges anywhere,

$$\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\epsilon\vec{E}),$$

$$= \vec{E}\cdot(\vec{\nabla}\epsilon) + \epsilon\vec{\nabla}\cdot\vec{E} = 0 . \tag{2.5}$$

The permittivity only depends on r, so we can write $\vec{\nabla}\epsilon = \dfrac{\mathrm{d}\epsilon}{\mathrm{d}r}\,\hat{e}_r$. Then putting this result into Eq. (2.5) with $\vec{E} = -\vec{\nabla}V$, we find

$$0 = (\vec{\nabla}V)\cdot\hat{e}_r\frac{\mathrm{d}\epsilon}{\mathrm{d}r} + \epsilon\nabla^2 V$$

$$= \frac{\partial V}{\partial r}\frac{\mathrm{d}\epsilon}{\mathrm{d}r}\frac{1}{\epsilon} + \nabla^2 V$$

$$\implies \boxed{0 = \frac{\partial V}{\partial r}\frac{\mathrm{d}\ln\epsilon}{\mathrm{d}r} + \nabla^2 V .} \tag{2.6}$$

(b) With an external field along the z-axis, the problem has azimuthal symmetry, implying $\partial V/\partial\phi = 0$, so $V = V(r,\theta)$. The Legendre polynomials $P_\ell(\cos\theta)$ are a complete set of functions of the polar angle θ for $0 \le \theta \le \pi$, implying that at each value of r, $V(r,\theta)$ can be expanded in a Legendre series. In general, the coefficients may be functions of r, so we can write

$$V(r,\theta) = \sum_{\ell=0}^{\infty} V_\ell(r)P_\ell(\cos\theta) . \tag{2.7}$$

The same argument holds for an expansion in $\{ \hat{z}_{i_1} \dots \hat{z}_{i_\ell} \} \hat{r}_{i_1} \dots \hat{r}_{i_\ell}$, since these are in fact the same functions, up to a multiplicative constant. Note that if ϵ depended on θ as well as r, then the completeness argument would still be valid, and it would still be possible to write $V(r, \theta)$ as in Eqs. (2.3). In that case, however, the equations for the functions $V_\ell(r)$ would become coupled to each other, making them much more difficult to solve.

(c) For $r < R$ we have $\dfrac{\mathrm{d} \ln \epsilon}{\mathrm{d}r} = -\dfrac{2}{r}$. Using the hint, Eq. (2.4) in the problem statement, we write

$$\nabla^2 V + \frac{\partial V}{\partial r} \frac{\mathrm{d} \ln \epsilon}{\mathrm{d}r} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V_\ell}{\partial r} \right) + \frac{\mathrm{d}V_\ell}{\mathrm{d}r} \left(-\frac{2}{r} \right) - \frac{\ell(\ell+1)}{r^2} V_\ell \right] = 0 \ . \tag{2.8}$$

For this equation to hold for all $r < R$ and for all θ, the term inside the square brackets should be zero. (To show this, one would multiply by $P_{\ell'}(\cos\theta)\sin\theta$ and then integrate from $\theta = 0$ to $\theta = 2\pi$. By the orthonormality of the Legendre polynomials, only the $\ell' = \ell$ term would survive, so it would have to vanish for every ℓ'.) Thus,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V_\ell}{\partial r} \right) + \frac{\mathrm{d}V_\ell}{\mathrm{d}r} \left(-\frac{2}{r} \right) - \frac{\ell(\ell+1)}{r^2} V_\ell = \frac{\mathrm{d}^2 V_\ell}{\mathrm{d}r^2} - \frac{\ell(\ell+1)}{r^2} V_\ell = 0 \ . \tag{2.9}$$

The general solution to Eq. (2.9) is

$$V_\ell(r) = A_\ell r^{\ell+1} + \frac{B_\ell}{r^\ell} \ . \tag{2.10}$$

(This can be verified by inspection, but it can also be found by assuming a trial function in the form of a power, $V_\ell \propto r^p$. Inserting the trial function into the differential equation, one finds $p(p-1) = \ell(\ell+1)$. One might see by inspection that this is solved by $p = \ell + 1$ or $p = -\ell$, or one can solve it as a quadratic equation, finding

$$p = \frac{1 \pm (2\ell+1)}{2} = \ell + 1 \text{ or } -\ell \ .)$$

For $r > R$,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V_\ell}{\partial r} \right) - \frac{\ell(\ell+1)}{r^2} V_\ell = 0. \tag{2.11}$$

The general solution to Eq. (2.11) is,

$$V_\ell(r) = C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}. \tag{2.12}$$

(d) The coefficients B_ℓ are zero, $B_\ell = 0$, to avoid a singularity at $r = 0$. The potential goes as $V(\vec{r}) = -E_0 r \cos\theta$ for $r \gg R$; this gives $C_\ell = 0$ except for $C_1 = -E_0$. The potential $V(r, \theta)$ is continuous at $r = R$, implying that

$$\begin{cases} A_\ell R^{\ell+1} = \dfrac{D_\ell}{R^{\ell+1}} & \text{for } \ell \neq 1 \\[3mm] A_1 R^2 = -E_0 R + \dfrac{D_1}{R^2} & \text{for } \ell = 1 \ . \end{cases} \tag{2.13}$$

In addition, the normal component of the displacement vector is continuous on the boundary of the sphere. Since ϵ is continuous at $r = R$, this means that $E_r = -\partial V/\partial r$ is continuous, which one could also have deduced from Eq. (2.2), since any discontinuity in $\partial V/\partial r$ would produce a δ-function in $\partial^2 V/\partial r^2$. Setting $\partial V/\partial r$ at $r = R^-$ equal to its value at $r = R^+$, we find

$$\begin{cases} (\ell+1)A_\ell R^\ell = -(\ell+1)\dfrac{D_\ell}{R^{\ell+2}} & \text{for } \ell \neq 1 \\[3mm] 2A_1 R = -2\dfrac{D_1}{R^3} - E_0 & \text{for } \ell = 1 \ . \end{cases} \tag{2.14}$$

Solving Eq. (2.13) and Eq. (2.14) as two equations (for each ℓ) for the two unknowns A_ℓ and D_ℓ, we see that $A_\ell = D_\ell = 0$ for $\ell \neq 1$, and that

$$A_1 = -\frac{3E_0}{4R} \ , \qquad C_1 = -E_0 \ , \quad \text{and} \quad D_1 = \frac{E_0 R^3}{4} \ . \tag{2.15}$$

Then we find the potential as

$$V(r, \theta) = \begin{cases} -\dfrac{3E_0 r^2}{4R} \cos\theta & \text{for } r < R \\[3mm] E_0 \cos\theta \left(\dfrac{R^3}{4r^2} - r \right) & \text{for } r < R \ . \end{cases} \tag{2.16}$$

(e) Eq. (2.16) tells us that for $r > R$, the potential is equal to that of the applied external field, $V_{\text{ext}} = -E_0 r \cos\theta$, plus a term that we attribute to the sphere:

$$V_{\text{sphere}}(r, \theta) = \frac{E_0 R^3}{4r^2} \cos\theta \ . \tag{2.17}$$

This has exactly the form of an electric dipole,

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} \ , \tag{2.18}$$

if we identify

$$\vec{p} = \pi\epsilon_0 R^3 E_0 \, \hat{z} \ . \tag{2.19}$$

PROBLEM 3: PAIR OF MAGNETIC DIPOLES *(20 points)*

Suppose there are two magnetic dipoles. One has dipole moment $\vec{m}_1 = m_0\hat{z}$ and is located at $\vec{r}_1 = +\frac{1}{2}a\,\hat{z}$; the other has dipole moment $\vec{m}_2 = -m_0\hat{z}$, and is located at $\vec{r}_2 = -\frac{1}{2}a\,\hat{z}$.

(a) *(10 points)* For a point on the z axis at large z, find the leading (in powers of $1/z$) behavior for the vector potential $\vec{A}(0,0,z)$ and the magnetic field $\vec{B}(0,0,z)$.

(b) *(3 points)* In the language of monopole ($\ell = 0$), dipole ($\ell = 1$), quadrupole ($\ell = 2$), octupole ($\ell = 3$), etc., what type of field is produced at large distances by this current configuration? In future parts, the answer to this question will be called a whatapole.

(c) *(3 points)* We can construct an ideal whatapole — a whatapole of zero size — by taking the limit as $a \to 0$, keeping $m_0 a^n$ fixed, for some power n. What is the correct value of n?

(d) *(4 points)* Given the formula for the current density of a dipole,

$$\vec{J}_{\rm dip}(\vec{r}) = -\vec{m} \times \vec{\nabla}_{\vec{r}}\,\delta^3(\vec{r} - \vec{r}_d) \;, \tag{3.1}$$

where $\vec{r}_d$ is the position of the dipole, find an expression for the current density of the whatapole constructed in part (c). Like the above equation, it should be expressed in terms of δ-functions and/or derivatives of δ-functions, and maybe even higher derivatives of δ-functions.

PROBLEM 3 SOLUTION:

(a) For the vector potential, we have from the formula sheet that

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2} \;, \tag{3.2}$$

which vanishes on axis, since $\vec{m} = m_0\hat{z}$, and $\hat{r} = \hat{z}$ on axis. Thus,

$$\vec{A}(0,0,z) = 0 \;. \tag{3.3}$$

This does not mean that $\vec{B} = 0$, however, since B depends on derivatives of $\vec{A}$ with respect to x and y. From the formula sheet we have

$$\vec{B}_{\rm dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} \;, \tag{3.4}$$

where we have dropped the δ-function because we are interested only in $r \neq 0$. Evaluating this expression on the positive z axis, where $\hat{r} = \hat{z}$, we find

$$\vec{B}_{\rm dip}(0,0,z) = \frac{\mu_0}{4\pi}\frac{2m_0\hat{z}}{r^3} = \frac{\mu_0}{2\pi}\frac{m_0\hat{z}}{r^3} \;. \tag{3.5}$$

For 2 dipoles, we have

$$\vec{B}_{2 \text{ dip}}(0,0,z) = \frac{\mu_0 m_0}{2\pi} \left[\frac{1}{\left(z - \frac{1}{2}a\right)^3} - \frac{1}{\left(z + \frac{1}{2}a\right)^3} \right] \hat{z}$$

$$= \frac{\mu_0 m_0}{2\pi z^3} \left[\frac{1}{\left(1 - \frac{1}{2}\frac{a}{z}\right)^3} - \frac{1}{\left(1 + \frac{1}{2}\frac{a}{z}\right)^3} \right] \hat{z}$$

$$\approx \frac{\mu_0 m_0}{2\pi z^3} \left[\frac{1}{\left(1 - \frac{3}{2}\frac{a}{z}\right)} - \frac{1}{\left(1 + \frac{3}{2}\frac{a}{z}\right)} \right] \hat{z}$$

$$\approx \frac{\mu_0 m_0}{2\pi z^3} \left[\left(1 + \frac{3}{2}\frac{a}{z}\right) - \left(1 - \frac{3}{2}\frac{a}{z}\right) \right] \hat{z}$$

$$\approx \frac{\mu_0 m_0}{2\pi z^3} \left[3\frac{a}{z} \right] \hat{z}$$

$$= \boxed{\frac{3\mu_0 m_0 a}{4\pi z^4} \hat{z}} . \tag{3.6}$$

(b) Since it falls off as $1/z^4$, it is undoubtedly a $\boxed{\text{quadrupole } (\ell = 2)}$. For either the $\vec{E}$ or $\vec{B}$ fields, the monopole falls off as $1/r^2$, the dipole as $1/r^3$, and the quadrupole as $1/r^4$.

(c) We wish to take the limit as $a \to 0$ in such a way that the field at large z approaches a constant, without blowing up or going to zero. From Eq. (3.6), we see that this goal will be accomplished by keeping $m_0 a$ fixed, which means $\boxed{n = 1}$.

(d) For the two-dipole system we add together the two contributions to the current density, using the appropriate values of $\vec{r}_d$ and $\vec{m}$:

$$\vec{J}_{2 \text{ dip}}(\vec{r}) = -m_0 \hat{z} \times \vec{\nabla}_{\vec{r}} \delta^3 \left(\vec{r} - \frac{a}{2}\hat{z}\right) + m_0 \hat{z} \times \vec{\nabla}_{\vec{r}} \delta^3 \left(\vec{r} - \frac{a}{2}\hat{z}\right) . \tag{3.7}$$

Rewriting,

$$\vec{J}_{2 \text{ dip}}(\vec{r}) = m_0 a \hat{z} \times \vec{\nabla}_{\vec{r}} \left[\frac{\delta^3(\vec{r} + \frac{a}{2}\hat{z}) - \delta^3(\vec{r} - \frac{a}{2}\hat{z})}{a} \right] . \tag{3.8}$$

Now we can define $Q \equiv m_0 a$, and if we take the limit $a \to 0$ with Q fixed, the above expression becomes

$$\boxed{\vec{J}_{2 \text{ dip}}(\vec{r}) = Q\hat{z} \times \vec{\nabla}_{\vec{r}} \frac{\partial}{\partial z} \delta^3(\vec{r})} . \tag{3.9}$$

Since partial derivatives commute, this could alternatively be written as

$$\vec{J}_{2\ \text{dip}}(\vec{r}) = Q\hat{z} \times \frac{\partial}{\partial z}\vec{\nabla}_{\vec{r}}\delta^3(\vec{r}) \ . \tag{3.10}$$

PROBLEM 4: UNIFORMLY MAGNETIZED INFINITE CYLINDER *(10 points)*

Consider a uniformly magnetized infinite circular cylinder, of radius R, with its axis coinciding with the z axis. The magnetization inside the cylinder is $\vec{M} = M_0\hat{z}$.

(a) *(5 points)* Find $\vec{H}(\vec{r})$ everywhere in space.

(b) *(5 points)* Find $\vec{B}(\vec{r})$ everywhere in space.

PROBLEM 4 SOLUTION:

(a) The magnetization inside the cylinder is $\vec{M} = M_0\hat{z}$. The curl of the $\vec{H}(\vec{r})$ field is

$$\vec{\nabla} \times \vec{H}(\vec{r}) = \vec{J}_{\text{free}} = 0 \ , \tag{4.1}$$

and the divergence is

$$\vec{\nabla} \cdot \vec{H}(\vec{r}) = \vec{\nabla} \cdot \left(\frac{\vec{B}(\vec{r})}{\mu_0} - \vec{M}(\vec{r}) \right) = \frac{1}{\mu_0}\vec{\nabla} \cdot \vec{B} - \vec{\nabla} \cdot \vec{M} = 0 \ . \tag{4.2}$$

Note that for a finite length cylinder, the divergence would be nonzero because of the abrupt change in $\vec{M}$ at the boundaries. Since $\vec{H}(\vec{r})$ is divergenceless and curl-free, we can say

$$\vec{H}(\vec{r}) = 0 \qquad \text{everywhere in space.} \tag{4.3}$$

(b) Having $\vec{H}(\vec{r}) = 0$ everywhere in space, we can find magnetic field as

$$\vec{H}(\vec{r}) = \frac{\vec{B}(\vec{r})}{\mu_0} - \vec{M}(\vec{r}) = 0 \quad \Longrightarrow \quad \vec{B}(\vec{r}) = \begin{cases} \mu_0 M_0\hat{z} & \text{for } r < R \ , \\ 0 & \text{for } r > R \ . \end{cases} \tag{4.4}$$

In this question we could alternatively find the bound currents as $\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$ and $\vec{K}_b = \vec{M} \times \hat{n} = M_0\hat{\phi}$. Then, using Ampère's law as we did for a solenoid, we could find the magnetic field and then also $\vec{H}$, obtaining the same answers as above.

## PROBLEM 5: ELECTRIC AND MAGNETIC UNIFORMLY POLARIZED SPHERES *(10 points)*

Compare the electric field of a uniformly polarized sphere with the magnetic field of a uniformly magnetized sphere; in each case the dipole moment per unit volume points along $\hat{z}$. Multiple choice: which of the following is true?

(a) The $\vec{E}$ and $\vec{B}$ field lines point in the same direction both inside and outside the spheres.

(b) The $\vec{E}$ and $\vec{B}$ field lines point in the same direction inside the spheres but in opposite directions outside.

(c) The $\vec{E}$ and $\vec{B}$ field lines point in opposite directions inside the spheres but in the same direction outside.

(d) The $\vec{E}$ and $\vec{B}$ field lines point in opposite directions both inside and outside the spheres.

PROBLEM 5 SOLUTION:
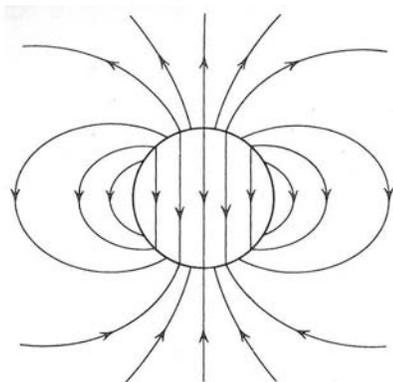
$\vec{E}$ field of a uniformly
polarized sphere
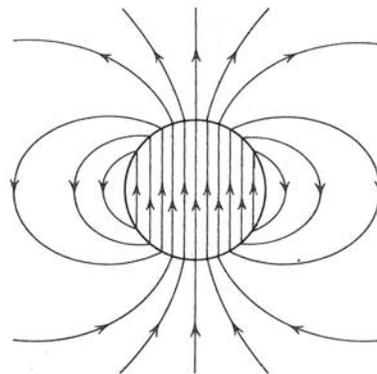
$\vec{B}$ field of a uniformly
magnetized sphere

The answer is (c), $\vec{E}$ and $\vec{B}$ field lines point in opposite directions inside the spheres but in the same direction outside, as shown in the diagrams, which were scanned from the first edition of Jackson. Note that the diagram on the left shows clearly that $\vec{\nabla} \cdot \vec{E} \neq 0$ at the boundary of the sphere, so it could not possibly be a picture of $\vec{B}$. It is at least visually consistent with $\vec{\nabla} \times \vec{E} = 0$, or equivalently $\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = 0$ for any closed loop, as it must be to describe an electrostatic field. The diagram on the right, on the other hand, shows clearly that $\vec{\nabla} \times \vec{B} \neq 0$, or equivalently $\oint \vec{B} \cdot \mathrm{d}\vec{\ell} \neq 0$, so it could not possibly be a picture of an electrostatic field. It is at least qualitatively consistent with $\vec{\nabla} \cdot \vec{B} = 0$, as it must be.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II   November 13, 2012
Prof. Alan Guth

FORMULA SHEET FOR QUIZ 2, V. 2
Exam Date: November 15, 2012

*** Some sections below are marked with asterisks, as this section is. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

Index Notation:

$$\vec{A} \cdot \vec{B} = A_i B_i\,, \qquad \vec{A} \times \vec{B}_i = \epsilon_{ijk} A_j B_k\,, \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n}$$

Rotation of a Vector:

$$A'_i = R_{ij} A_j\,, \qquad \text{Orthogonality: } R_{ij} R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

Rotation about z-axis by ϕ:
$$R_z(\phi)_{ij} = \begin{array}{c} i=1 \\ i=2 \\ i=3 \end{array} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(with column labels $j=1$, $j=2$, $j=3$)

Rotation about axis $\hat{n}$ by ϕ:***

$$R(\hat{n}, \phi)_{ij} = \delta_{ij}\cos\phi + \hat{n}_i \hat{n}_j (1 - \cos\phi) - \epsilon_{ijk}\hat{n}_k \sin\phi\,.$$

Vector Calculus:

Gradient: $(\vec{\nabla}\varphi)_i = \partial_i \varphi\,, \qquad \partial_i \equiv \dfrac{\partial}{\partial x_i}$

Divergence: $\vec{\nabla} \cdot \vec{A} \equiv \partial_i A_i$

Curl: $(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk}\partial_j A_k$

Laplacian: $\nabla^2 \varphi = \vec{\nabla} \cdot (\vec{\nabla}\varphi) = \dfrac{\partial^2 \varphi}{\partial x_i \partial x_i}$

Fundamental Theorems of Vector Calculus:

Gradient: $\displaystyle\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\varphi \cdot \mathrm{d}\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$

Divergence: $\displaystyle\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{A}\,\mathrm{d}^3 x = \oint_S \vec{A} \cdot \mathrm{d}\vec{a}$

where S is the boundary of $\mathcal{V}$

Curl: $\displaystyle\int_S (\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{a} = \oint_P \vec{A} \cdot \mathrm{d}\vec{\ell}$

where P is the boundary of S

Delta Functions:

$$\int \varphi(x)\delta(x - x')\,\mathrm{d}x = \varphi(x')\,, \qquad \int \varphi(\vec{r})\delta^3(\vec{r} - \vec{r}')\,\mathrm{d}^3x = \varphi(\vec{r}')$$

$$\int \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x - x')\,\mathrm{d}x = -\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}\,, \quad g(x_i) = 0$$

$$\vec{\nabla}\cdot\left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\right) = -\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = 4\pi\delta^3(\vec{r} - \vec{r}')$$

$$\partial_i\left(\frac{\hat{r}_j}{r^2}\right) \equiv \partial_i\left(\frac{x_j}{r^3}\right) = -\partial_i\partial_j\left(\frac{1}{r}\right) = \frac{\delta_{ij} - 3\hat{r}_i\hat{r}_j}{r^3} + \frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{r})$$

$$\vec{\nabla}\cdot\frac{3(\vec{d}\cdot\hat{r})\hat{r} - \vec{d}}{r^3} = -\frac{8\pi}{3}(\vec{d}\cdot\vec{\nabla})\delta^3(\vec{r})$$

$$\vec{\nabla}\times\frac{3(\vec{d}\cdot\hat{r})\hat{r} - \vec{d}}{r^3} = -\frac{4\pi}{3}\vec{d}\times\vec{\nabla}\delta^3(\vec{r})$$

Electrostatics:

$$\vec{F} = q\vec{E}\,, \text{ where}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\vec{r} - \vec{r}')\,q_i}{|\vec{r} - \vec{r}'|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}\rho(\vec{r}')\,\mathrm{d}^3x'$$

$$\epsilon_0 = \text{permittivity of free space} = 8.854\times10^{-12}\ \mathrm{C}^2/(\mathrm{N\cdot m}^2)$$

$$\frac{1}{4\pi\epsilon_0} = 8.988\times10^9\ \mathrm{N\cdot m^2/C^2}$$

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}')\cdot\mathrm{d}\vec{\ell}' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}\mathrm{d}^3x'$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}\,, \qquad \vec{\nabla}\times\vec{E} = 0\,, \qquad \vec{E} = -\vec{\nabla}V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}\ \text{ (Poisson's Eq.)}\,, \qquad\qquad \rho = 0 \implies \nabla^2 V = 0\ \text{ (Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

Energy:

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij \\ i\neq j}}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$W = \frac{1}{2}\int \mathrm{d}^3x\,\rho(\vec{r})V(\vec{r}) = \frac{1}{2}\epsilon_0\int \left|\vec{E}\right|^2\mathrm{d}^3x$$

Conductors:

Just outside, $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{n}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j \,, \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij} V_j \,, \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\dfrac{a}{R}Q$, $r = \dfrac{a^2}{R}$

Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \left\{ \begin{array}{c} \cos \alpha x \\ \sin \alpha x \end{array} \right\} \left\{ \begin{array}{c} \cos \beta y \\ \sin \beta y \end{array} \right\} \left\{ \begin{array}{c} \cosh \gamma z \\ \sinh \gamma z \end{array} \right\} \quad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

Separation of Variables for Laplace's Equation in Spherical Coordinates:

Traceless Symmetric Tensor expansion:

$$\nabla^2 \varphi(r, \theta, \phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\nabla_\theta^2 \varphi = 0 \,,$$

where the angular part is given by

$$\nabla_\theta^2 \varphi \equiv \frac{1}{\sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{\sin^2 \theta}\frac{\partial^2 \varphi}{\partial \phi^2}$$

$$\nabla_\theta^2 C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} = -\ell(\ell+1) C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \,,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a symmetric traceless tensor and

$$\hat{n} = \sin \theta \cos \phi \, \hat{e}_1 + \sin \theta \sin \phi \, \hat{e}_2 + \cos \theta \, \hat{e}_3 \,.$$

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}} \right) \hat{r}_{i_1} \hat{r}_{i_2} \ldots \hat{r}_{i_\ell} \,, \quad \text{where } \vec{r} = r\hat{r}$$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \{ \, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \, \} \, \hat{r}_{i_1} \ldots \hat{r}_{i_\ell}$$

where $\{ \ldots \}$ denotes the traceless symmetric part of $\ldots$.

Special cases:

$$\{ \, 1 \, \} = 1$$

$$\{ \, \hat{z}_i \, \} = \hat{z}_i$$

$$\{ \, \hat{z}_i \hat{z}_j \, \} = \hat{z}_i \hat{z}_j - \tfrac{1}{3} \delta_{ij}$$

$$\{ \, \hat{z}_i \hat{z}_j \hat{z}_k \, \} = \hat{z}_i \hat{z}_j \hat{z}_k - \tfrac{1}{5} \left(\hat{z}_i \delta_{jk} + \hat{z}_j \delta_{ik} + \hat{z}_k \delta_{ij} \right)$$

$$\{ \, \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \, \} = \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m - \tfrac{1}{7} \left(\hat{z}_i \hat{z}_j \delta_{km} + \hat{z}_i \hat{z}_k \delta_{mj} + \hat{z}_i \hat{z}_m \delta_{jk} + \hat{z}_j \hat{z}_k \delta_{im} \right.$$
$$\left. + \hat{z}_j \hat{z}_m \delta_{ik} + \hat{z}_k \hat{z}_m \delta_{ij} \right) + \tfrac{1}{35} \left(\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk} \right)$$

Legendre Polynomial / Spherical Harmonic expansion:

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} \, r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

Orthonormality: $\displaystyle \int_0^{2\pi} \mathrm{d}\phi \int_0^\pi \sin\theta \, \mathrm{d}\theta \, Y^*_{\ell' m'}(\theta, \phi) \, Y_{\ell m}(\theta, \phi) = \delta_{\ell' \ell} \delta_{m' m}$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Electric Multipole Expansion:

First several terms:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{1}{2} \frac{\hat{r}_i \hat{r}_j}{r^3} Q_{ij} + \cdots \right] \quad , \text{ where}$$

$$Q = \int \mathrm{d}^3x \, \rho(\vec{r}), \quad p_i = \int \mathrm{d}^3x \, \rho(\vec{r}) \, x_i \quad Q_{ij} = \int \mathrm{d}^3x \, \rho(\vec{r})(3x_i x_j - \delta_{ij} |\vec{r}|^2),$$

$$\vec{E}_{\text{dip}}(\vec{r}) = -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{\vec{p} \cdot \hat{r}}{r^2} \right) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{r})$$

$$\vec{\nabla} \times \vec{E}_{\text{dip}}(\vec{r}) = 0 , \qquad \vec{\nabla} \cdot \vec{E}_{\text{dip}}(\vec{r}) = \frac{1}{\epsilon_0} \rho_{\text{dip}}(\vec{r}) = -\frac{1}{\epsilon_0} \vec{p} \cdot \vec{\nabla} \delta^3(\vec{r})$$

Traceless Symmetric Tensor version:

$$
V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1 \dots i_\ell} \hat{r}_{i_1} \dots \hat{r}_{i_\ell} \ ,
$$

where

$$
C^{(\ell)}_{i_1 \dots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}) \left\{ x_{i_1} \dots x_{i_\ell} \right\} \mathrm{d}^3 x \qquad (\vec{r} \equiv r\hat{r} \equiv x_i \hat{e}_i)
$$

$$
\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \left\{ \hat{r}_{i_1} \dots \hat{r}_{i_\ell} \right\} \hat{r}'_{i_1} \dots \hat{r}'_{i_\ell} \ , \qquad \text{for } r' < r
$$

$$
(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\dots 1 = \frac{(2\ell)!}{2^\ell \ell!} \ , \ \text{with} \ (-1)!! \equiv 1 \ .
$$

Reminder: $\left\{ \dots \right\}$ denotes the traceless symmetric part of $\dots$.

Griffiths version:

$$
V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^\ell \rho(\vec{r}') P_\ell(\cos\theta') \mathrm{d}^3 x
$$

where $\theta' = $ angle between $\vec{r}$ and $\vec{r}'$.

$$
\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta') \ , \qquad \frac{1}{\sqrt{1-2\lambda x + \lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)
$$

$$
P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (x^2-1)^\ell \ , \qquad \text{(Rodrigues' formula)}
$$

$$
P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^{1} \mathrm{d}x \, P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1} \delta_{\ell'\ell}
$$

Spherical Harmonic version:***

$$
V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta,\phi)
$$

where $q_{\ell m} = \displaystyle\int Y_{\ell m}^* r'^\ell \rho(\vec{r}') \, \mathrm{d}^3 x'$

$$
\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^\ell}{r^{\ell+1}} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi) \ , \qquad \text{for } r' < r
$$

Electric Fields in Matter:

Electric Dipoles:

$$\vec{p} = \int d^3x \, \rho(\vec{r}) \, \vec{r}$$

$\rho_{\text{dip}}(\vec{r}) = -\vec{p} \cdot \vec{\nabla}_{\vec{r}} \, \delta^3(\vec{r} - \vec{r}_d)$, where $\vec{r}_d$ = position of dipole

$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} = \vec{\nabla}(\vec{p} \cdot \vec{E})$ (force on a dipole)

$\vec{\tau} = \vec{p} \times \vec{E}$ (torque on a dipole)

$U = -\vec{p} \cdot \vec{E}$

Electrically Polarizable Materials:

$\vec{P}(\vec{r})$ = polarization = electric dipole moment per unit volume

$\rho_{\text{bound}} = -\nabla \cdot \vec{P}$, $\sigma_{\text{bound}} = \vec{P} \cdot \hat{n}$

$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$, $\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$, $\vec{\nabla} \times \vec{E} = 0$ (for statics)

Boundary conditions:

$$E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = \frac{\sigma}{\epsilon_0} \qquad D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \sigma_{\text{free}}$$

$$\vec{E}_{\text{above}}^{\parallel} - \vec{E}_{\text{below}}^{\parallel} = 0 \qquad \vec{D}_{\text{above}}^{\parallel} - \vec{D}_{\text{below}}^{\parallel} = \vec{P}_{\text{above}}^{\parallel} - \vec{P}_{\text{below}}^{\parallel}$$

Linear Dielectrics:

$\vec{P} = \epsilon_0 \chi_e \vec{E},$ χ_e = electric susceptibility

$\epsilon \equiv \epsilon_0(1 + \chi_e)$ = permittivity, $\vec{D} = \epsilon \vec{E}$

$\epsilon_r = \dfrac{\epsilon}{\epsilon_0} = 1 + \chi_e$ = relative permittivity, or dielectric constant

Clausius-Mossotti equation: $\chi_e = \dfrac{N\alpha/\epsilon_0}{1 - \frac{N\alpha}{3\epsilon_0}}$, where N = number density of atoms

     or (nonpolar) molecules, α = atomic/molecular polarizability $(\vec{P} = \alpha \vec{E})$

Energy: $W = \dfrac{1}{2} \displaystyle\int \vec{D} \cdot \vec{E} \, \mathrm{d}^3 x$ (linear materials only)

Force on a dielectric: $\vec{F} = -\vec{\nabla}W$ (Even if one or more potential differences are held fixed, the force can be found by computing the gradient with the total charge on each conductor fixed.)

Magnetostatics:

Magnetic Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \, , \qquad \text{where } \vec{p} = \gamma m_0 \vec{v} \, , \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec{F} = \int I \mathrm{d}\vec{\ell} \times \vec{B} = \int \vec{J} \times \vec{B} \, \mathrm{d}^3 x$$

Current Density:

Current through a surface S: $I_S = \displaystyle\int_S \vec{J} \cdot \mathrm{d}\vec{a}$

Charge conservation: $\dfrac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$

Moving density of charge: $\vec{J} = \rho \vec{v}$

Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\ell'} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, \mathrm{d}a'$$

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, \mathrm{d}^3 x$$

where μ_0 = permeability of free space $\equiv 4\pi \times 10^{-7}$ N/A^2

Examples:

Infinitely long straight wire: $\vec{B} = \dfrac{\mu_0 I}{2\pi r} \, \hat{\phi}$

Infintely long tightly wound solenoid: $\vec{B} = \mu_0 n I_0 \, \hat{z}$, where n = turns per unit length

Loop of current on axis: $\vec{B}(0,0,z) = \dfrac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \, \hat{z}$

Infinite current sheet: $\vec{B}(\vec{r}) = \dfrac{1}{2}\mu_0 \vec{K} \times \hat{n}$, $\hat{n}$ = unit normal toward $\vec{r}$

Vector Potential:

$$\vec{A}(\vec{r})_{\text{coul}} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, \mathrm{d}^3 x' \ , \qquad \vec{B} = \vec{\nabla} \times \vec{A} \ , \qquad \vec{\nabla} \cdot \vec{A}_{\text{coul}} = 0$$

$\vec{\nabla} \cdot \vec{B} = 0$ (Subject to modification if magnetic monopoles are discovered)

Gauge Transformations: $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Lambda(\vec{r})$ for any $\Lambda(\vec{r})$. $\vec{B} = \vec{\nabla} \times \vec{A}$ is unchanged.

Ampère's Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \ , \ \text{ or equivalently } \int_P \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enc}}$$

Magnetic Multipole Expansion:

Traceless Symmetric Tensor version:

$$A_j(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \mathcal{M}^{(\ell)}_{j;i_1 i_2 \ldots i_\ell} \frac{\{\hat{r}_{i_1} \ldots \hat{r}_{i_\ell}\}}{r^{\ell+1}}$$

where $\mathcal{M}^{(\ell)}_{j;i_1 i_2 \ldots i_\ell} = \dfrac{(2\ell-1)!!}{\ell!} \displaystyle\int \mathrm{d}^3 x J_j(\vec{r}) \{ x_{i_1} \ldots x_{i_\ell} \}$

Current conservation restriction: $\displaystyle\int \mathrm{d}^3 x \operatorname*{Sym}_{i_1 \ldots i_\ell} (x_{i_1} \ldots x_{i_{\ell-1}} J_{i_\ell}) = 0$

where Sym means to symmetrize — i.e. average over all
$\quad{}_{i_1 \ldots i_\ell}$
　　orderings — in the indices $i_1 \ldots i_\ell$

Special cases:

$\ell = 1$: $\quad \displaystyle\int \mathrm{d}^3 x\, J_i = 0$

$\ell = 2$: $\quad \displaystyle\int \mathrm{d}^3 x\, (J_i x_j + J_j x_i) = 0$

Leading term (dipole): $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi} \dfrac{\vec{m} \times \hat{r}}{r^2}$,

where

$$m_i = -\frac{1}{2} \epsilon_{ijk} \mathcal{M}^{(1)}_{j;k}$$

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times \mathrm{d}\vec{\ell} = \frac{1}{2} \int \mathrm{d}^3 x\, \vec{r} \times \vec{J} = I\vec{a} ,$$

where $\vec{a} = \displaystyle\int_S \mathrm{d}\vec{a}$ for any surface S spanning P

$$\vec{B}_{\mathrm{dip}}(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3} \vec{m}\, \delta^3(\vec{r})$$

$$\vec{\nabla} \cdot \vec{B}_{\mathrm{dip}}(\vec{r}) = 0 , \qquad \vec{\nabla} \times \vec{B}_{\mathrm{dip}}(\vec{r}) = \mu_0 \vec{J}_{\mathrm{dip}}(\vec{r}) = -\mu_0 \vec{m} \times \vec{\nabla} \delta^3(\vec{r})$$

Griffiths version:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \oint (r')^\ell P_\ell(\cos\theta') \mathrm{d}\vec{\ell'}$$

Magnetic Fields in Matter:

Magnetic Dipoles:

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times \mathrm{d}\vec{\ell} = \frac{1}{2} \int \mathrm{d}^3 x\, \vec{r} \times \vec{J} = I\vec{a}$$

$\vec{J}_{\text{dip}}(\vec{r}) = -\vec{m} \times \vec{\nabla}_{\vec{r}} \delta^3(\vec{r} - \vec{r}_d)$, where $\vec{r}_d = $ position of dipole

$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$ (force on a dipole)

$\vec{\tau} = \vec{m} \times \vec{B}$ (torque on a dipole)

$U = -\vec{m} \cdot \vec{B}$

Magnetically Polarizable Materials:

$\vec{M}(\vec{r}) = $ magnetization = magnetic dipole moment per unit volume

$\vec{J}_{\text{bound}} = \vec{\nabla} \times \vec{M}$, $\vec{K}_{\text{bound}} = \vec{M} \times \hat{n}$

$\vec{H} \equiv \dfrac{1}{\mu_0}\vec{B} - \vec{M}$, $\vec{\nabla} \times \vec{H} = \vec{J}_{\text{free}}$, $\vec{\nabla} \cdot \vec{B} = 0$

Boundary conditions:

$B_{\text{above}}^{\perp} - B_{\text{below}}^{\perp} = 0$ $H_{\text{above}}^{\perp} - H_{\text{below}}^{\perp} = -(M_{\text{above}}^{\perp} - M_{\text{below}}^{\perp})$

$\vec{B}_{\text{above}}^{\parallel} - \vec{B}_{\text{below}}^{\parallel} = \mu_0(\vec{K} \times \hat{n})$ $\vec{H}_{\text{above}}^{\parallel} - \vec{H}_{\text{below}}^{\parallel} = \vec{K}_{\text{free}} \times \hat{n}$

Linear Magnetic Materials:

$\vec{M} = \chi_m \vec{H}$, $\chi_m = $ magnetic susceptibility

$\mu = \mu_0(1 + \chi_m) = $ permeability, $\vec{B} = \mu \vec{H}$

Magnetic Monopoles:

$\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}\dfrac{q_m}{r^2}\hat{r}$;   Force on a static monopole: $\vec{F} = q_m \vec{B}$

Angular momentum of monopole/charge system: $\vec{L} = \dfrac{\mu_0 q_e q_m}{4\pi}\hat{r}$, where $\hat{r}$ points from q_e to q_m

Dirac quantization condition: $\dfrac{\mu_0 q_e q_m}{4\pi} = \dfrac{1}{2}\hbar \times$ integer

Connection Between Traceless Symmetric Tensors and Legendre Polynomials or Spherical Harmonics:

$P_\ell(\cos\theta) = \dfrac{(2\ell)!}{2^\ell(\ell!)^2}\{\, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \,\}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$

For $m \geq 0$,

$Y_{\ell m}(\theta, \phi) = C_{i_1 \ldots i_\ell}^{(\ell, m)}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$,

where $C_{i_1 i_2 \ldots i_\ell}^{(\ell, m)} = d_{\ell m}\{\, \hat{u}_{i_1}^+ \ldots \hat{u}_{i_m}^+ \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \,\}$,

with $d_{\ell m} = \dfrac{(-1)^m (2\ell)!}{2^\ell \ell!}\sqrt{\dfrac{2^m (2\ell + 1)}{4\pi (\ell + m)!\,(\ell - m)!}}$,

and $\hat{u}^+ = \dfrac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$

Form $m < 0$, $Y_{\ell, -m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi)$

More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} \, P_\ell^m(\cos\theta) e^{im\phi}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} (1 - x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^\ell$$

Legendre Polynomials:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

SPHERICAL HARMONICS $Y_{lm}(\theta, \varphi)$

$l = 0$ $\quad Y_{00} = \dfrac{1}{\sqrt{4\pi}}$

$l = 1$

$\quad Y_{11} = -\sqrt{\dfrac{3}{8\pi}} \sin\theta e^{i\varphi}$

$\quad Y_{10} = \sqrt{\dfrac{3}{4\pi}} \cos\theta$

$l = 2$

$\quad Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}} \sin^2\theta e^{2i\varphi}$

$\quad Y_{21} = -\sqrt{\dfrac{15}{8\pi}} \sin\theta \cos\theta e^{i\varphi}$

$\quad Y_{20} = \sqrt{\dfrac{5}{4\pi}} \left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$

$l = 3$

$\quad Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}} \sin^3\theta e^{3i\varphi}$

$\quad Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}} \sin^2\theta \cos\theta e^{2i\varphi}$

$\quad Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}} \sin\theta (5\cos^2\theta - 1) e^{i\varphi}$

$\quad Y_{30} = \sqrt{\dfrac{7}{4\pi}} \left(\tfrac{5}{2}\cos^3\theta - \tfrac{3}{2}\cos\theta\right)$

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8.07 Electromagnetism II
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