# Magnetic Dipoles

11/7/12 Lecture ~~25~~ of 37

Magnetic multipole expansion for a localized charge distribution

Recall

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^{\ell}}{r^{\ell+1}}$$

$$\times \{\hat{r}'_{i_1} \dots \hat{r}'_{i_\ell}\} \{\hat{r}_{i_1} \dots \hat{r}_{i_\ell}\}$$

where $r' < r$, $r \equiv |\vec{r}|$, $\vec{r} \equiv x_i \hat{e}_i$, $\hat{r}_i \equiv \frac{x_i}{r}$

$(2\ell-1)!! = (2\ell-1)(2\ell-3)\dots(1) = \frac{(2\ell)!}{2^\ell \ell!}$, $(-1)!! \equiv 1$

and $\{\ \}$ $\equiv$ traceless symmetric part

$$\{1\} = 1$$
$$\{\hat{r}_i\} = \hat{r}_i$$
$$\{\hat{r}_i \hat{r}_j\} = \hat{r}_i \hat{r}_j - \tfrac{1}{3}\delta_{ij}$$
$$\dots$$

(Posted as Lecture Notes 9)

Combine with

$$A_j(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J_j(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3x'$$

For $|\vec{r}| >$ radius of charge distribution, expand $\frac{1}{|\vec{r}-\vec{r}\,'|}$ as above:

$$\boxed{A_j(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \mathcal{M}^{(\ell)}_{j;i_1\dots i_\ell} \frac{\{\hat{r}_{i_1}\dots\hat{r}_{i_\ell}\}}{r^{\ell+1}}}$$

where

$$\boxed{\mathcal{M}^{\ell}_{j;i_1\dots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int d^3x\, J_j(\vec{r}) \{x_{i_1}\dots x_{i_\ell}\}}$$

I dropped primes in integral.

Note: $r'^{\ell}\{\hat{r}'_{i_1}\dots\hat{r}'_{i_\ell}\} = \{x'_{i_1}\dots x'_{i_\ell}\}$

Restrictions on $\mathcal{M}^{\ell}_{j;i_1\dots i_\ell}$ due to current conservation: $\partial_i J_i = 0$

$$\int d^3x\, \partial_j J_j\, x_{i_1}\dots x_{i_\ell} = 0$$

int by parts

$$= -\int d^3x\, J_j\, \partial_j (x_{i_1}\dots x_{i_\ell})$$

$$= -\int d^3x\, J_j\, (\delta_{i_1 j}\, x_{i_2}\dots x_{i_\ell} + \delta_{i_2 j}\, x_{i_1} x_{i_3}\dots x_{i_\ell} + \dots)$$

$$= \quad -\int d^3x \, ( J_{i_1} x_{i_2} \cdots x_{i_\ell} + x_{i_1} J_{i_2} x_{i_3} \cdots x_{i_\ell} + \cdots )$$

$$= -\ell \int d^3x \, \underset{i_1 \cdots i_\ell}{\mathrm{Sym}} \, ( x_{i_1} \cdots x_{i_{\ell-1}} J_{i_\ell} )$$

$\underset{i_1 \cdots i_\ell}{\mathrm{Sym}}$ $\equiv$ Sum over all $\ell!$ permutations and divide by $\ell!$

$\equiv$ Average over permutations.

Conclusion:

$$\boxed{\int d^3x \, \underset{i_1 \cdots i_\ell}{\mathrm{Sym}} \, ( x_{i_1} \cdots x_{i_{\ell-1}} J_{i_\ell} ) = 0}$$

$\ell = 1$: $\quad \int d^3x \, J_i = 0$

$\ell = 2$: $\quad \int d^3x \, ( J_i x_j + J_j x_i ) = 0$

Trace: $\quad \delta_{ij} \int d^3x \, ( J_i x_j + J_j x_i ) = 0$

$$\Rightarrow \quad \int d^3x \, ( \vec{r} \cdot \vec{J} ) = 0$$

$\ell = 3$: $\quad \int d^3x \, ( J_i x_j x_k + J_j x_i x_k + J_k x_i x_j ) = 0$

Trace: $\quad \delta_{jk} \int \cdots = 0 \Rightarrow \int d^3x \, [ J_i r^2 + 2 x_i ( \vec{J} \cdot \vec{r} ) ] = 0$

$$\mathcal{M}^{(\ell)}_{j;\,i_1\ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!}\int d^3x\, J_j\,\{x_{i_1}\ldots x_{i_\ell}\}$$

$\ell=0$: $$\mathcal{M}^{(0)}_j \propto \int d^3x\, J_j = 0$$

$\ell=1$: $$\mathcal{M}^{(1)}_{j;i} = \int d^3x\, J_j\, x_i$$

Current constraint: Symmetric part vanishes. Must be antisymmetric:

$$\mathcal{M}^{(1)}_{i;j} = -\mathcal{M}^{(1)}_{j;i}$$

Antisymmetric tensor is equiv. to vector, using $\epsilon_{ijk}$.

Define dipole moment vector $\vec{m}$,

$$m_i = \tfrac{1}{2}\epsilon_{ijk}\int d^3x\, x_j\, J_k$$

$$\boxed{\vec{m} = \frac{1}{2}\int d^3x\, \vec{r}\times\vec{J}}$$

$$m_i = \frac{1}{2}\epsilon_{ijk}\,\mathcal{M}^{(1)}_{k;j}$$

$$\mathcal{M}^{(1)}_{j;i} = m_k\,\epsilon_{kij}$$

Pf:

$$m_k \epsilon_{kij} = \epsilon_{kij} \tfrac{1}{2} \epsilon_{kmn} \mathcal{M}^{(1)}_{n,m}$$

$$= \tfrac{1}{2} \left[ \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \right] \mathcal{M}^{(1)}_{n,m}$$

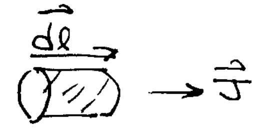
$$= \mathcal{M}^{(1)}_{j;i}$$

$$A_j(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \mathcal{M}^{(\ell)}_{j;i_1\ldots i_\ell} \frac{\{\hat{r}_{i_1}\ldots\hat{r}_{i_\ell}\}}{r^{\ell+1}}$$

$$= \frac{\mu_0}{4\pi} \Big[ \underset{\uparrow \atop \ell=0}{0} + m_k \epsilon_{kij} \frac{\hat{r}_i}{r^2} + \ldots \Big]$$

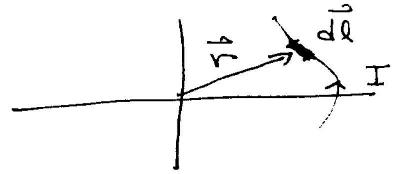
$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m}\times\hat{r}}{r^2} + \ldots}$$

Currents in wires:




$$d^3x\, \vec{J} \longrightarrow I\, \vec{d\ell}$$

$$\vec{m} = \tfrac{1}{2} I \int \vec{r} \times \vec{d\ell}$$



Another form:

Recall: $\int_P \vec{V}\cdot \vec{d\ell} = \int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a}$


$$\int_P V_i \, d\ell_i = \int_S \epsilon_{ijk} (\partial_j V_k)\, da_i$$

$$\int_P [\vec{r} \times \vec{d\ell}]_m = \int_P \epsilon_{mnp}\, x_n \, d\ell_p$$

$$= \int_P \underbrace{(\epsilon_{imn} x_n)}_{V_i} d\ell_i$$

$$= \int_S \epsilon_{ijk} [\partial_j (\epsilon_{kmn} x_n)]\, da_i$$

$$= \int_S \epsilon_{ijk}\epsilon_{kmn}\delta_{jn} \; da_i$$

$$= \int_S \epsilon_{ijk}\epsilon_{kmj} \; da_i$$

$$= \int_S \underbrace{\epsilon_{ijk}\,\epsilon_{mjk}}_{2\delta_{im}} \; da_i$$

$$= 2\int_S da_m$$

$$\frac{1}{2}\int_P (\vec{r}\times d\vec{\ell}) = \int d\vec{a} = \text{vector area of } S. = \vec{a}$$

$$\boxed{\vec{m} = I\vec{a}}$$

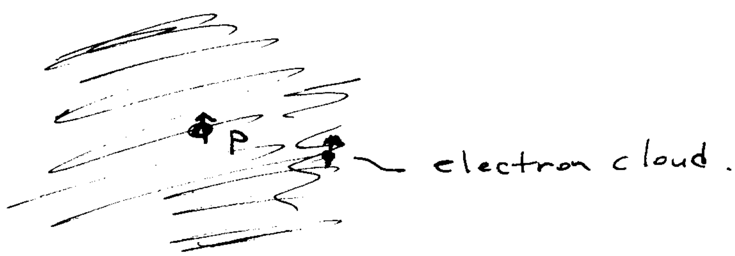
For a loop in a plane, $|\vec{a}|$ = area, direction $\perp$ to plane.

Magnetic field of a dipole (PS 7, Prob 7)

$$\boxed{\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3}\vec{m}\,\delta^3(\vec{r})}$$

Significance of $\delta^3(\vec{r})$ term:

Hydrogen orbital ground state:




Orbit is $l=0$ (angular momentum $= 0$ — spherically symmetric ~~etc~~ probability density).

Spins interact by dipole-dipole interaction.

2-states: aligned + antialigned

Depends on average $\vec{B}_{proton}$ experienced by electron.

Transition is astronomically crucial.
Galaxy is mapped by 21 cm line.

$\Delta E$ for transition comes 100% from $\delta$-function term.

Force on a magnetic dipole –

Recall, electric dipole:

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} \quad \text{or} \quad \vec{\nabla}(\vec{p}\cdot\vec{E})$$

$$\vec{\tau} = \vec{p}\times\vec{E} + \vec{r}\times\vec{F}$$

$$U = -\vec{p}\cdot\vec{E}$$

~~For~~ Magnetic dipoles –

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

$$\vec{\tau} = \vec{m}\times\vec{B} + \vec{r}\times\vec{F}$$

$$U = -\vec{m}\cdot\vec{B}.$$

Magnetization:

$\vec{M}$ = magnetic dipole moment per unit volume

Bound currents:

$$\vec{J}_b = \vec{\nabla}\times\vec{M} \qquad \text{volume current}$$

$$\vec{K}_b = \vec{M}\times\hat{n} \qquad \text{surface current}$$

$\hat{n}$ = outward normal.

$\vec{K}$ = current per cross sectional length flowing on surface.

Maxwell's equations in matter (magnetostatics).

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

Now write ~~$I = I_{free} + I_b$~~

$$\vec{J} = \vec{J}_f + \vec{J}_b$$
$$= \vec{J}_f + \vec{\nabla}\times\vec{M}$$

Define $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

$$\vec{\nabla}\times\vec{H} = \vec{J}_{free}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

In linear materials,

$$\vec{M} = \chi_m \vec{H}$$