VIII The Fermi gas:

① Free Fermions:

✱ Single particle states $|\vec{k}\rangle$ $\quad \vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$

Multi-particle states

$$|n_{k_1}, n_{k_2} \cdots \rangle \equiv |\{n_{\vec{k}}\}\rangle$$

$n_{\vec{k}}$ is # of particles on $|\vec{k}\rangle$

For fermions $n_{\vec{k}} = 0, 1$.

Energy of state $|\{n_{\vec{k}}\}\rangle$

$$\boxed{E(\{n_{\vec{k}}\}) = \sum_k \epsilon_k n_k} \qquad \epsilon_k = \frac{\hbar^2 k^2}{2m}$$

✱ Grand partition function

$$Q_G = \sum_{\{n_k\}} e^{-\beta(\sum \epsilon_k n_k - \mu N)} \qquad N = \sum n_k$$

$$= \sum_{\{n_k\}} e^{-\beta \sum (\epsilon_k - \mu) n_k}$$

$$= \prod_k \sum_{n_k} e^{-\beta (\epsilon_k - \mu) n_k}$$

$$= \prod_k \left[1 + e^{-\beta(\epsilon_k - \mu)}\right]$$

Thermopotential:

$$\boxed{\Omega = -k_B T \ln Q_G = -k_B T \sum_k \ln\left(1 + e^{-\beta(\epsilon_k - \mu)}\right)}$$

☆ Total # of particle

$$N = -\frac{\partial \Omega}{\partial \mu}\bigg|_{V,T} = \sum_k \frac{e^{-\beta(\epsilon_k-\mu)}}{1+e^{-\beta(\epsilon_k-\mu)}}$$

$$= \sum_k \frac{1}{1+e^{\beta(\epsilon_k-\mu)}}$$

# of particle in state $|\vec{k}\rangle$

$$\boxed{n_k = \frac{1}{1+e^{\beta(\epsilon_k-\mu)}}} \qquad N = \sum_k n_k = V\int \frac{d^3k}{(2\pi)^3}\frac{1}{1+e^{\beta(\epsilon_k-\mu)}}$$

Fermi-Dirac distribution

☆ Equation of state.

$$P = -\frac{\partial \Omega}{\partial V}\bigg|_{\mu,T} = -\sum_k \frac{\frac{\partial \epsilon_k}{\partial V}e^{-\beta(\epsilon_k-\mu)}}{1+e^{-\beta(\epsilon_k-\mu)}}$$

$$= -\sum_k \frac{\partial \epsilon_k}{\partial V} n_k \qquad \epsilon_k = \frac{\hbar^2 k^2}{2m}$$

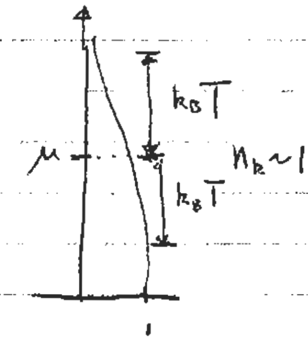
$$= \frac{2}{3}\frac{1}{V}\sum_k \epsilon_k n_k \qquad = \frac{\hbar^2(n_x^2+n_y^2+n_z^2)\frac{(2\pi)^2}{V^{2/3}}}{2m}$$

$$= \frac{2}{3}\frac{1}{V}L^3\int\frac{d^3k}{(2\pi)^3}\frac{\hbar^2k^2}{2m}\frac{1}{1+e^{\beta(\epsilon_k-\mu)}} \qquad \frac{\partial \epsilon_k}{\partial V} = -\frac{2}{3}\frac{1}{V}\epsilon_k$$

$$\boxed{P = \frac{2}{3}\int\frac{d^3k}{(2\pi)^3}\frac{\hbar^2k^2}{2m}\frac{1}{1+e^{\beta(\epsilon_k-\mu)}}} \qquad \boxed{\#\text{ of states} = d^3k\frac{L^3}{(2\pi)^3}}$$

$$\boxed{n = \frac{N}{V} = \int\frac{d^3k}{(2\pi)^3}\frac{1}{1+e^{\beta(\epsilon_k-\mu)}}} \Rightarrow \boxed{\begin{aligned} PV &= \frac{2}{3}U(T,V,\mu) \\ U &= V\int\frac{d^3k}{(2\pi)^3}\epsilon_k n_k\end{aligned}}$$

$$\Rightarrow \mu(N,T,V) \Rightarrow PV = \frac{2}{3}U(T,V,\mu(T,V,N))$$



★ High temperature (classical) limit

n fixed $\quad T \to \infty \quad \mu \to ?$

to have fixed n $\quad \mu \to -\infty$

and $n_k \propto \frac{1}{T} \to 0$

$$P = \frac{2}{3} \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} e^{-\beta\epsilon_k} e^{+\beta\mu}$$

$$n = \int \frac{d^3k}{(2\pi)^3} e^{-\beta\epsilon_k} e^{+\beta\mu}$$

$$\frac{P}{n} = \frac{\frac{2}{3}\int d^3k\, \epsilon_k e^{-\beta\epsilon_k}}{\int d^3k\, e^{-\beta\epsilon}} = \frac{2}{3}\langle\epsilon_k\rangle \qquad I(\beta) = \int d^3k\, e^{-\beta\epsilon_k} \propto \beta^{-3/2}$$

$$= \frac{2}{3} \frac{-\frac{\partial}{\partial\beta} I(\beta)}{I(\beta)} = \frac{1}{\beta} = k_B T \qquad \boxed{PV = \frac{2}{3}U} \quad \boxed{U = N\langle\epsilon_k\rangle}$$

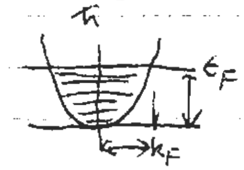
$$PV = N k_B T \qquad \text{classical gas eq. of state}$$

At high temperature $\boxed{\text{Fermi gas} = \text{classical gas}}$

★ Zero-temperature (quantum) limit

$T = 0 \qquad n_k = 1 \quad$ if $\epsilon_k < \mu = \epsilon_F$ or $k < k_F = \frac{\sqrt{2m\mu}}{\hbar}$

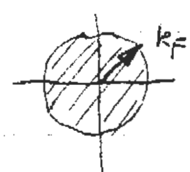
$\qquad\qquad n_k = 0 \quad$ if $\epsilon_k > \mu = \epsilon_F$



$$P = \frac{2}{3} \int_{k<k_F} \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} = \frac{2}{3}\frac{\hbar^2}{(2\pi)^3 2m} \int_0^{k_F} 4\pi k^4\, dk$$

$$= \frac{2}{3}\frac{\hbar^2}{(2\pi)^3 2m} \frac{4\pi}{5} k_F^5$$

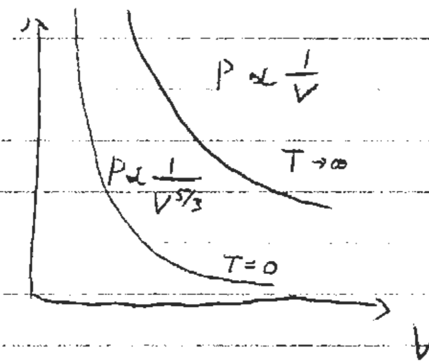
$$= \frac{\hbar^2}{30\pi^2 m} k_F^5$$



$$n = \int \frac{d^3k}{(2\pi)^3} = \frac{1}{(2\pi)^3}\int_0^{k_F} 4\pi k^2 dk$$

$$= \frac{1}{(2\pi)^3}\,\frac{4\pi}{3}\,k_F^3 = \frac{1}{6\pi^2}\,k_F^3$$

$$\boxed{k_F = (6\pi^2 n)^{1/3}} \qquad \boxed{\epsilon_F = \frac{\hbar^2 k_F^2}{2m}}$$



$$P = \frac{\hbar^2}{30\pi^2 m}(6\pi^2 n)^{5/3}$$

$$= \frac{\hbar^2 (6\pi^2)^{5/3}}{30\pi^2\, m}\, n^{5/3}$$

$$\boxed{P = \frac{(6\pi^2)^{5/3}}{15\pi^2}\,\frac{\hbar^2 n^{2/3}}{2m}\, n}$$

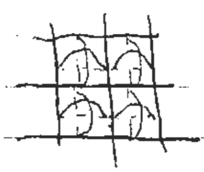
$$\boxed{V^{5/3} P = \frac{(6\pi^2)^{5/2}\hbar^2}{30\pi^2\, m}\, N^{5/3} = \text{const.}}$$

* Quantum limit and classical limit

$T = 0$ $\quad P \sim$ energy per volum $\quad (E \sim PV)$

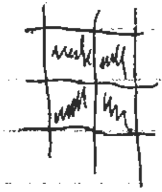
$= $ energy per particle $\times\, n$

$\sim \dfrac{\hbar^2 n^{2/3}}{2m} \times n$



↑ Fermi pressure

high temperature (classical)



classical limit
$$\boxed{\begin{aligned} &\frac{\sqrt{2m k_B T}}{\hbar} \ll n^{1/3} \\ &\sim \; k_B T \gg \frac{\hbar^2 k_F^2}{2m} \equiv \epsilon_F \end{aligned}}$$

★ High temperature expansion — correction to ideal gas.

Free energy at high temperatures

$$\Omega = -k_B T \sum_k \ln\left(1 + e^{-\beta(\epsilon_k - \mu)}\right)$$

$$= -k_B T V \int \frac{d^3k}{(2\pi)^3} \ln\left(1 + e^{-\beta(\epsilon_k - \mu)}\right)$$

$$A = \Omega + \mu N \Big|_{\mu = \mu(V,N,T)}$$

$$N = -\frac{\partial \Omega}{\partial \mu} = V \int \frac{d^3k}{(2\pi)^3} \underbrace{\frac{1}{1 + e^{\beta(\epsilon_k - \mu)}}}_{= \frac{z e^{-\beta\epsilon_k}}{1 + z e^{-\beta\epsilon_k}}}$$

solve for $\mu(V,N,T)$

In high temperature $e^{-\beta(\epsilon_k - \mu)} = z e^{-\beta\epsilon_k} \ll 1$

$z = e^{\beta\mu}$

expand to second order in "z"

$$N = V \int \frac{d^3k}{(2\pi)^3} \left(z e^{-\beta\epsilon_k} - z^2 e^{-2\beta\epsilon_k}\right)$$

$$= V\left(z \lambda^{-3} - z^2 2^{-3/2} \lambda^{-3}\right)$$

$$\int \frac{d^3k}{(2\pi)^3} e^{-\beta\epsilon_k} = \left(\sqrt{2\pi\hbar^2/mk_BT}\right)^{-3} = \lambda^{-3}$$

$$\Rightarrow n\lambda^3 = z - z^2/2^{3/2}$$

$$\simeq z - \frac{(n\lambda^3)^2}{2^{3/2}}$$

$$z = n\lambda^3 + \frac{(n\lambda^3)^2}{2^{3/2}}$$

$$\mu = k_B T \ln z$$

$$A = -k_B T V \int \frac{d^3k}{(2\pi)^3} \left(z e^{-\beta \epsilon_k} - \frac{1}{2} z^2 e^{-2\beta\epsilon_k} \right)$$

$$+ k_B T \left[\ln n\lambda^3 + \ln\left(1 + \frac{n\lambda^3}{2^{3/2}}\right) \right] N$$

$$= -k_B T V \frac{1}{\lambda^3} \underbrace{\left(n\lambda^3 + \frac{(n\lambda^3)^2}{2^{3/2}} \right)}_{z}$$

$$+ \frac{1}{2} k_B T V \frac{1}{2^{3/2}\lambda^3} \left(n\lambda^3 + \frac{(n\lambda^3)^2}{2^{3/2}} \right)^2$$

$$+ k_B T \left(\ln n\lambda^3 + \frac{n\lambda^3}{2^{3/2}} \right) N$$

$$= -k_B T N \left(1 + \frac{n\lambda^3}{2^{3/2}} \right)$$

$$+ \frac{1}{2} k_B T N \frac{n\lambda^3}{2^{3/2}} + k_B T \left(\ln n\lambda^3 + \frac{n\lambda^3}{2^{3/2}} \right) N$$

$$\boxed{A = k_B T N \left(\ln n\lambda^3 - 1 + \frac{1}{2} \frac{n\lambda^3}{2^{3/2}} \right)}$$

quantum correction, small if $n\lambda^3 \ll 1$

Eqn. of state

$$A = k_B T N \left(-\ln V + \frac{1}{2} \frac{N\lambda^3}{2^{3/2} V} \right) + \dots$$

$$P = -\frac{\partial A}{\partial V} = k_B T / V + \underbrace{k_B T \frac{N^2 \lambda^3}{2^{5/2} V^2}}_{\text{extra pressure}}$$

Virial expansion

$$\frac{PV}{k_B T} = 1 + \frac{n\lambda^3}{2^{5/2}} = 1 + \frac{N\lambda^3/2^{5/2}}{V} = 1 + \frac{c_2}{V} + \frac{c_3}{V} \dots$$

$$c_2 = \frac{N\lambda^3}{2^{5/2}} > 0$$

✱ Low temperature properties:

Density of states:

$D(\epsilon)d\epsilon$ = # of states with energy between ϵ and $\epsilon + d\epsilon$

$$D(\epsilon) = V \int \frac{d^3k}{(2\pi)^3} \delta(\epsilon_k - \epsilon) \quad \text{(for 3D)}$$

$$N(\epsilon_F) = \text{\# of states below } \epsilon_F = \int_0^{\epsilon_F} D(\epsilon)\, d\epsilon$$

$$= V \int_{\substack{\epsilon_k < \epsilon_F \\ \text{or } k < k_F}} \frac{d^3k}{(2\pi)^3} = \frac{V}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = \boxed{\frac{V}{6\pi^2}\left(\frac{\sqrt{2m}}{\hbar}\right)^3 \epsilon_F^{3/2}}$$

$$k_F = \frac{\sqrt{2m\epsilon_F}}{\hbar}$$

$$\boxed{N(\epsilon_F) = \text{\# of fermions} = \frac{V}{6\pi^2} k_F^3}$$ (spinless, one fermion per state)

$$\boxed{k_F^3 \sim \text{number density}}$$

$$= 2 \times \frac{V}{6\pi^2} k_F^3$$ (spin-$\frac{1}{2}$, two fermion per state)

↑ from spin.

$$\boxed{D(\epsilon) = \frac{\partial N(\epsilon)}{\partial \epsilon} = \frac{\sqrt{2}}{2\pi^2} \frac{m^{3/2}}{\hbar^3} \epsilon^{1/2}}$$ 3D

Zero temperature:

Ground state energy

$$U_0 = \int_0^{\epsilon_F} d\epsilon \, \epsilon \, D(\epsilon)$$

$$= \frac{\sqrt{2}}{2\pi^2} \frac{m^{3/2}}{\hbar^3} V \int_0^{\epsilon_F} d\epsilon \, \epsilon \, \epsilon^{1/2}$$

$$= \frac{\sqrt{2}}{2\pi^2} \frac{m^{3/2}}{\hbar^3} V \frac{2}{5} \epsilon_F^{5/2}$$

$$N_0 = \int_0^{\epsilon_F} d\epsilon \, D(\epsilon)$$

$$= \frac{\sqrt{2}}{2\pi^2} \frac{m^{3/2}}{\hbar^3} V \int_0^{\epsilon_F} d\epsilon \, \epsilon^{1/2}$$

$$= \frac{\sqrt{2}}{2\pi^2} \frac{m^{3/2}}{\hbar^3} V \frac{2}{3} \epsilon_F^{3/2}$$

$$\boxed{U_0 = N \frac{3}{5} \epsilon_F}$$

energy per particle $\sim \epsilon_F$

We have shown that

$$\boxed{VP = -V \frac{\partial \epsilon_k}{\partial V} = \frac{2}{3} V \epsilon_k \qquad \epsilon_k \propto V^{-\frac{2}{3}}}$$

$$PV = \frac{2}{3} V \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \frac{1}{1 + e^{\beta(\epsilon_k - \mu)}}$$

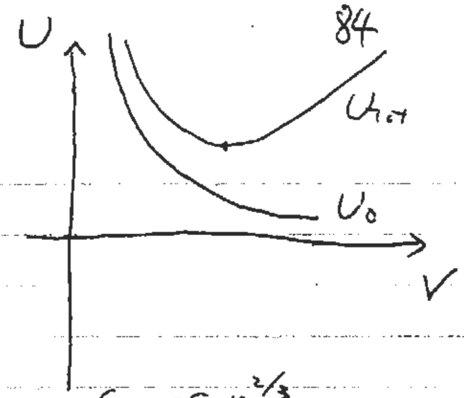
$$= \frac{2}{3} V \int \frac{d^3k}{(2\pi)^3} \epsilon_k n_k = \frac{2}{3} U$$

$$\boxed{P_0 = \frac{2}{3} \frac{U}{V} = \frac{2}{5} n \epsilon_F}$$

$n \to k_F \to \epsilon_F$

in metal $n \sim 10^{22}/\text{cm}^3$ $\quad P_0 \sim 10^4 \text{ atm}$

$\epsilon_F \sim$ a few eV $\gg k_B T \sim \frac{1}{40}$ eV



Compressibility of a metal

$$U_0 = N\frac{3}{5}\epsilon_F = C N n_e^{2/3} \quad \text{(} N_\uparrow + N_\downarrow \text{)}$$

$$= C N^{5/3} / V^{2/3}$$

$$C = \frac{3}{5}\frac{\hbar^2 (3\pi^2)^{2/3}}{2m}$$

$\epsilon_F \propto n^{2/3}$

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$$

$$n_{spin\,1/2} = 2 \cdot \frac{k_F^3}{6\pi^2}$$

$$n_{spin\,0} = \frac{k_F^3}{6\pi^2}$$

$$k_F = (3\pi^2 n_e)^{1/3}$$

$$\epsilon_F = \frac{\hbar^2 (3\pi^2)^{2/3}}{2m} n_e^{2/3}$$

Model

$$U_{tot} = \frac{C N^{5/3}}{V^{2/3}} + P_0 V$$

equ. volume

$$\frac{\partial U_{tot}}{\partial V} = 0 \quad \Rightarrow \quad P_0 = \frac{2}{3}\frac{C N^{5/3}}{V_0^{5/3}}$$

$$V_0 = \left(\frac{3 P_0}{2 C N^{5/3}}\right)^{3/5}$$

Compressibility

$$\chi \equiv -\frac{1}{V}\frac{dV}{dP} = \frac{-1}{V\frac{dP}{dV}} = \frac{1}{V\frac{\partial^2 U_{tot}}{\partial V^2}}$$

$$= \frac{1}{V\,\frac{2}{3}\,\frac{5}{3}\,\frac{C N^{5/3}}{V^{8/3}}} = \frac{9}{10}\frac{1}{C\, n^{5/3}}$$

$$\chi = 9\frac{m}{\hbar^2 (3\pi^2)^{2/3}}\frac{1}{n^{5/3}} = \frac{9}{2}\frac{1}{\epsilon_F n} \sim \frac{1}{10^4\,\text{atm}}$$

Need 10^4 atm to reduce $V \to \frac{1}{2} V$

1 atm ~ 10 m water → 10^5 m of water

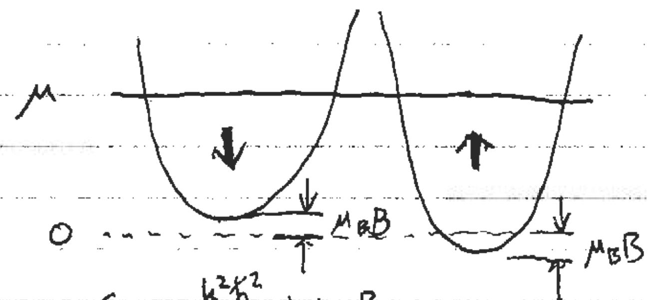
Spin susceptibility of metal.

magnetic moment $\uparrow$ μ_B $\downarrow$ $-\mu_B$

Induced $M = \mu_B (N_\uparrow - N_\downarrow)$

$$\Delta N = 2\mu_B B \, D(\epsilon_F)$$

$$= 2\mu_B B \frac{\sqrt{2}}{2\pi^2} \frac{m^{3/2}}{\hbar^3} \epsilon_F^{1/2} V$$



$$\epsilon_\downarrow = \frac{k^2\hbar^2}{2m} + \mu_B B$$

$$\epsilon_\uparrow = \frac{k^2\hbar^2}{2m} - \mu_B B$$

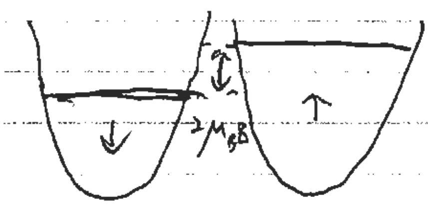
$$n = \frac{1}{6\pi^2} \left(\frac{\sqrt{2m}}{\hbar}\right)^3 \epsilon_F^{3/2} \times 2$$

($\times 2$: spin)

$$\propto \epsilon_F^{3/2}$$

$$\epsilon_F \propto n^{2/3}$$

$$\Delta N \propto n^{1/3}$$



$$\chi = \frac{M}{B} = 2\mu_B^2 \frac{\sqrt{2}}{(2\pi)^2} \frac{m^{3/2}}{\hbar^3} \epsilon_F^{1/2} V$$

$$= \mu_B^2 \frac{\sqrt{2}\, 3^{1/3}}{\pi} \frac{m}{\hbar^2} n^{1/3} V$$

Low temperature specific heat

$$U = \int d\epsilon \; D(\epsilon) \; \epsilon \, n_F(\epsilon) \qquad n_F(\epsilon) = \frac{1}{1+e^{\beta(\epsilon-\mu)}}$$

$$N = \int d\epsilon \; D(\epsilon) \; n_F(\epsilon) \quad \Rightarrow \text{ find } \mu = \mu(N,V,T)$$

$$U(\mu, V, T) \Rightarrow U[\mu(N,V,T), V, T]$$

$$C = \left.\frac{\partial U}{\partial T}\right|_{N,V}$$

$$\left.\frac{\partial N}{\partial T}\right|_{N,V} = \int d\epsilon \; D(\epsilon) \left.\frac{\partial n_F[\epsilon, T, \mu(N,V,T)]}{\partial T}\right|_{N,V} = 0$$

$$\Rightarrow \int d\epsilon \; \mu \, D(\epsilon) \left.\frac{\partial n_F}{\partial T}\right|_{N,V} = 0$$

$$\Rightarrow C = \int d\epsilon \; D(\epsilon) \; \epsilon \left.\frac{\partial n_F}{\partial T}\right|_{N,V}$$

$$= \int d\epsilon \; D(\epsilon) \; (\epsilon - \mu) \left.\frac{\partial n_F}{\partial T}\right|_{N,V}$$

$$\left.\frac{\partial n_F}{\partial T}\right|_{N,V} = \underbrace{\frac{\epsilon - \mu}{k_B T^2}}_{\to O(T^{-1})} \frac{e^{\beta(\epsilon-\mu)}}{[1+e^{\beta(\epsilon-\mu)}]^2}$$

$$\underbrace{+ \frac{1}{k_B T} \underbrace{\left.\frac{\partial \mu}{\partial T}\right|_{N,V}}_{\to \sim T} \frac{e^{\beta(\epsilon-\mu)}}{[1+e^{\beta(\epsilon-\mu)}]^2}}_{\to \sim O(T^0) \quad \text{droped}}$$

$$C = \frac{D(\mu)}{k_B T^2} \int d\epsilon \, (\epsilon - \mu)^2 \frac{e^{\beta(\epsilon-\mu)}}{[1 + e^{\beta(\epsilon-\mu)}]^2}$$

$$t = \beta(\epsilon - \mu)$$

$$= \frac{D(\mu)}{k_B T^2} k_B^3 T^3 \int_{-\infty}^{+\infty} dt \, t^2 \frac{e^t}{(1+e^t)^2}$$

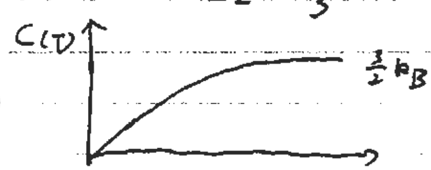
$$\boxed{C = \frac{\pi^2}{3} k_B^2 T \, D(\mu)}$$ works for ∀ dimensions

$$I_n = 2\int_0^\infty dt \frac{t^n e^t}{(1+e^t)^2}$$

$$D(\mu) \simeq \frac{3N}{2\mu}$$

$I_0 = 1 \quad I_2 = \frac{\pi^2}{3}$

$\mu = \epsilon_F$



$$\boxed{C = \frac{\pi^2}{2} k_B \frac{k_B T}{\epsilon_F} N} \quad \text{(3D)}$$

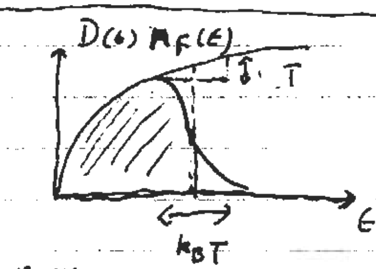
$$= \frac{3}{2} k_B \frac{\frac{\pi^2}{3} k_B T}{\epsilon_F} N$$

Classical gas. T

$$U = N \tfrac{3}{2} k_B T$$

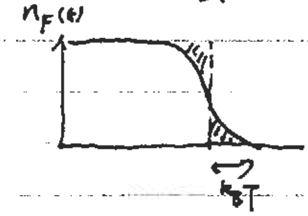
$$C = N \tfrac{3}{2} k_B$$

Why $\left.\frac{\partial \mu}{\partial T}\right|_{N,V} \sim T$



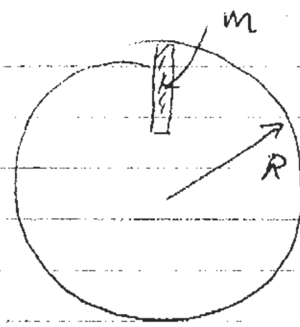
$$N = \int d\epsilon \, D(\epsilon) \, n_F(\epsilon)$$

$$= N_0 + * T^2$$



$$\Rightarrow \mu = \mu_0 + * T^2$$

White dwarf & neutron star



$$P \approx \frac{\hbar^2 n^{5/3}}{m}$$

$$P_{tot} = P_e + P_p \qquad P_p << P_e$$

$$\simeq P_e \simeq \frac{\hbar^2 n^{5/3}}{m_e}$$

$m_n = 939.565$ MeV
$m_p = 938.271$ MeV

Δm 1.3 MeV
$m_e = 0.51$ MeV

Balance:

$$P_{tot} \approx G \frac{Mm}{R^2}$$

$$= G \frac{M}{R^2} \underbrace{(n m_p R)}_{= m}$$

$$= G M m_p n / R = \frac{\hbar^2 n^{5/3}}{m_e}$$

$$n \simeq \frac{M}{m_p R^3}$$

$$G M m_p m_e / R = \hbar^2 \left(\frac{M}{m_p R^3}\right)^{2/3}$$

$$= \hbar^2 \frac{M^{2/3}}{m_p^{2/3} R^2}$$

$$R_{WD} = M^{-1/3} \frac{\hbar^2}{m_e m_p^{5/3}} G^{-1}$$

$$= \left(\frac{M_\odot}{M}\right)^{1/3} M_\odot^{-1/3} \frac{\hbar^2}{m_e m_p^{5/3} G}$$

$$\left[\frac{\hbar^2}{m^3 G}\right]$$

$$= [\hbar^2] [m^{-1} L^{-1}] \left[\frac{L}{m^2 G}\right]$$

$$= \left[\frac{\hbar^2}{m L E}\right] \checkmark$$

$$\boxed{R_{WD} = \left(\frac{M_\odot}{M}\right)^{1/3} 6200 \text{ km}}$$

$M_\odot \simeq 1.99 \times 10^{33}$ g

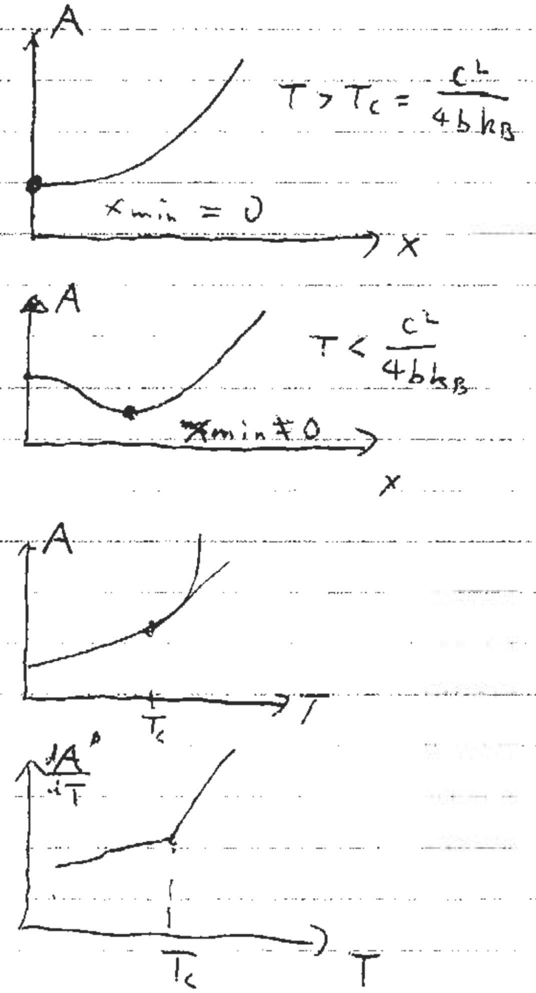
For neutron star we replace m_e by m_p:

$$\boxed{R_{NS} = \left(\frac{M_\odot}{M}\right)^{1/3} 3.4 \text{ km}}$$

Prob. 12.6

$$Q_1 = \sum \exp(-\beta\epsilon_n) = 2e^{\beta(-bx^2 - cx/2)} + e^{-\beta(bx^2+cx)}$$

$$A_{(x)} = -k_B T N \ln Q_1$$



Semi conductor.

Band theory

$\psi_k(x) = e^{ikx}$

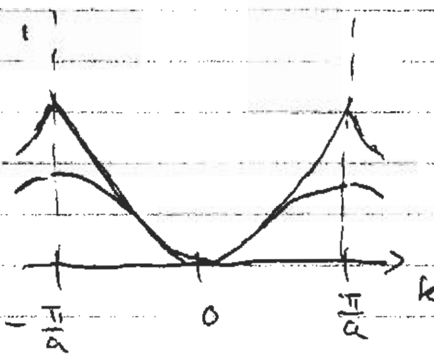
$$\boxed{\epsilon_k = \frac{k^2\hbar^2}{2m}}$$

on lattice $x = na$ $n = 0, \pm 1, \pm 2 \cdots$

$\psi_k(x) \rightarrow \psi_k(n) = e^{ikna}$ $\epsilon_k = \frac{k^2\hbar^2}{2m}$

But $\psi_{k+K}(n) = \psi_k$ if $K = \frac{2\pi}{a}$

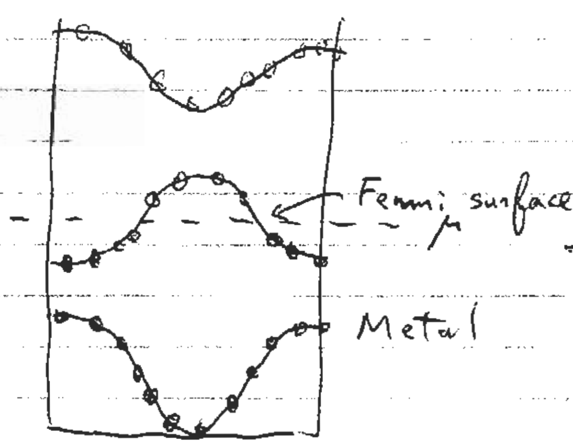
Brillouin zone

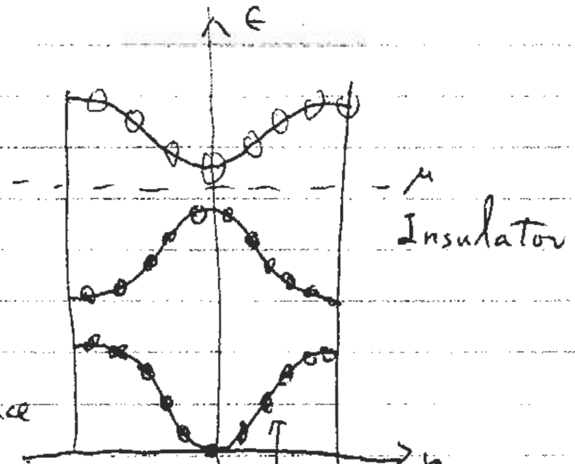


$$\boxed{\epsilon_k = \frac{\hbar^2}{ma^2}[1 - \cos(ka)]}$$

↑ for small k $\epsilon_k = \frac{k^2\hbar^2}{2m}$

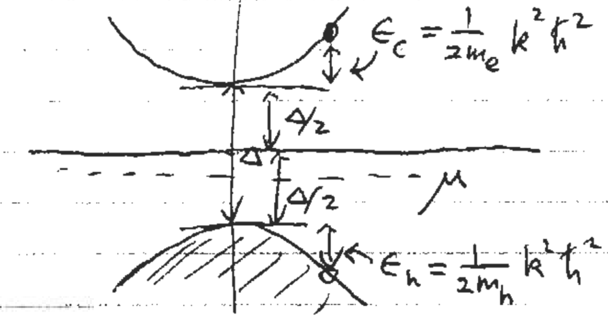
Band structure:





# of levels in each band
= # unit cell.

Method 1.



$$n_e = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon_e + \frac{\Delta}{2} - \mu)} + 1}$$

$$n_h = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon_h + \frac{\Delta}{2} + \mu)} + 1}$$

Adjust μ to make $n_e = n_h$

If $m_e = m_h \Rightarrow \mu = 0$

$$n_e = n_h = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon + \frac{\Delta}{2})} + 1}$$

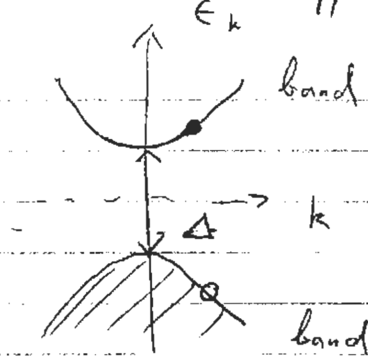
For large T ($k_B T \gg \Delta$)

$$n_e \simeq \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\epsilon} + 1} = \frac{1}{2\pi^2}\left(\frac{2m k_B T}{\hbar^2}\right)^{3/2} \int_0^\infty dt \frac{t^2}{e^t + 1}$$

$$\sim \left(\frac{m k_B T}{\hbar^2}\right)^{3/2} \sim \frac{1}{\lambda^3} \quad \text{(quantum)}$$

For small T ($k_B T \ll \Delta$)

$$n_e \simeq \int \frac{d^3k}{(2\pi)^3} e^{-\beta\epsilon} e^{-\beta\frac{\Delta}{2}} = \lambda^{-3} e^{-\beta\Delta/2} \quad \text{(classical)}$$



Method 2

$$\epsilon_e = \frac{1}{2m_c} k^2$$

$$\epsilon_h = \frac{1}{2m_v} k^2$$

$$(ele) + (hole) = \Delta$$

$$\nu_1 X_1 + \nu_2 X_2 + \cdots = \epsilon_0$$

$$\Rightarrow (\lambda_1^3 n_1)^{\nu_1} (\lambda_2^3 n_2)^{\nu_2} \cdots = e^{-\frac{\epsilon_0}{k_B T}}$$

$k_B T \ll \Delta$ $\quad$ $n_k \ll 1$ or $1 - n_k \ll 1$

classical gas of particles & holes

$$\lambda_e^3 n_e \lambda_h^3 n_h = e^{-\Delta/k_B T}$$

$$\lambda = \sqrt{2\pi \hbar^2 / m k_B T}$$

$\lambda^3 n \ll 1$
Classical
$\lambda^3 n \gtrsim 1$
Quantum

$$n_e = n_h = 2 e^{-\Delta/2k_B T} \left(\frac{\sqrt{m_e m_v}\, k_B T}{2\pi \hbar^2} \right)^{3/2}$$

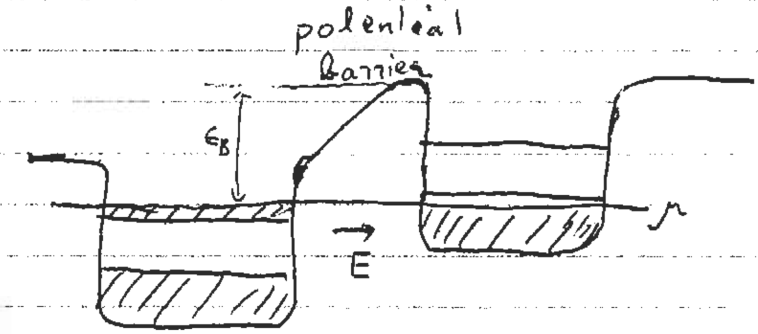
$$\simeq T^{3/2} e^{-\frac{\Delta}{2k_B T}}$$

Conductivity of semicoductor (no doping)

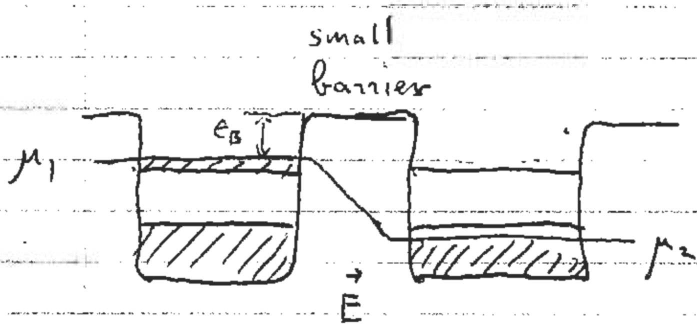
$$\sigma \propto n_e \sim n_h \propto T^{3/2} e^{-\frac{\Delta}{2k_B T}}$$

Diode (p-n junction):

n-type (extra electrons)   p-type (extra holes)

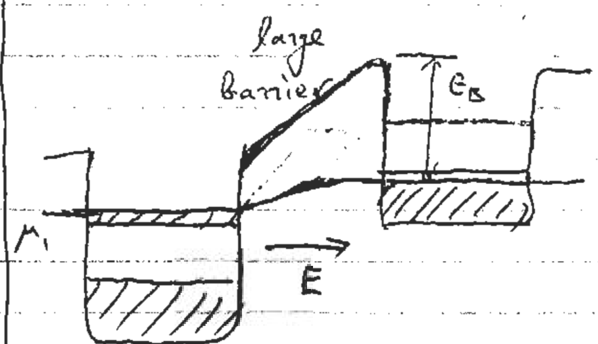


positive bias



→ electron flow

negative bias



← electron flow

$$I \sim V e^{-\epsilon_B / k_B T}$$