Classical Mechanics III (8.09) Fall 2014
Assignment 9

Massachusetts Institute of Technology
Physics Department
Mon. November 17, 2014

Due Wed. November 26, 2014
6:00pm

Announcements

This week we start our study of chaos.

I've made this problem set due on the Wednesday before Thanksgiving, so that you have 3 lectures on chaos before it is due. The material we will cover this week is sufficient to do these problems.

Reading Assignment

- We have finished our discussion of fluids, but there is one final problem on this Pset. Further reading on fluids, including lots of worked examples and plenty of problems, can be found in the book "A physical introduction to Fluid Mechanics" by **Alexander Smits**, which is on reserve in the physics reading room.

- Read the posted Chapter 3 of **Baker and Gollub** on the Nonlinear Damped Forced Oscillator.

- Read the posted sections from Chapter 3 of **Strogatz** on Bifurcations.

Problem Set 9

These 5 problems include one on fluids and four on chaos.

1. **Viscous Flow on an Inclined Plane** [10 points]

   Consider a thin layer of water of constant depth h, viscosity ν, and density ρ which flows down an infinite plane inclined at an angle θ to the horizontal, as shown below. The flow is steady, incompressible, and laminar (ie. not turbulent). It is acted upon by gravity. On the surface open to the air take the pressure $p = p_{atm}$.
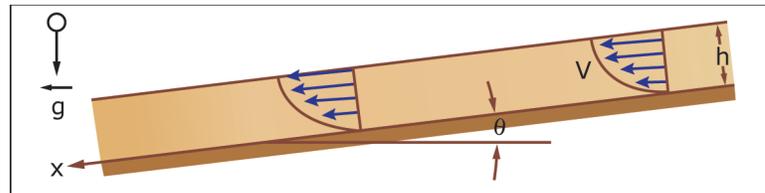
   Image by MIT OpenCourseWare.

   (a) [2 points] What is an appropriate form of the Navier-Stokes equation for this problem?

   (b) [4 points] What does symmetry tell us about the velocity $\vec{v}$? What does it tell us about the pressure p? What are appropriate boundary conditions for $\hat{x} \cdot \vec{v}$ on the surface of the inclined plane and on the surface that is open to the air?

   (c) [4 points] Solve the Navier-Stokes equation to determine $\vec{v}$ and p in the fluid.

2. **Damped Nonlinear Oscillator** [15 points]

   Consider the damped nonlinear oscillator with equations of motion

$$\dot{\theta} = w \ , \qquad\qquad\qquad \dot{w} = -\frac{1}{q} w - \sin\theta$$

   where $q > 0$ is the quality factor. Recall that the fixed points have $w = 0$ and $\theta = n\pi$ for any integer n.

   (a) [2 points] To analyze the behavior of trajectories near a fixed point we can use linear differential equations obtained by expanding the nonlinear terms about that fixed point. Find these equations for the fixed points of the damped nonlinear oscillator.

   (b) [4 points] Lets start by considering the case $q = \infty$. You either have elliptical oscillations around the fixed point, or the fixed point is a saddle point. Derive solutions to your equations from (a) for these two possibilities. For the saddle point determine the directions in the (w, θ) phase space that correspond to purely growing or purely decaying solutions.

For the last two parts consider the case with damping, so use your equations from (a) for finite $q > 0$. The fixed points are either attractors or saddle points.

(c) [4 points] Near an attractor fixed point derive solutions that determine the phase space trajectory for $q > 1/2$ (underdamped) and $q < 1/2$ (overdamped). Roughly sketch a trajectory for each case.

(d) [5 points] Find the solution near a saddle point and identify growing and decaying modes. What is the exponential growth rate (the $\kappa > 0$ in $e^{\kappa t}$)? What is the angle between the direction of a purely growing solution and the θ axis?

3. **Lorenz Equations** [10 points]

Lorenz considered the following equations as a model for convection in the atmosphere

$$\dot{x} = \sigma y - \sigma x$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -bz + xy$$

where $\sigma > 0$, $r > 0$, $b > 0$ are constant parameters. (x is related to the fluid flow, and y and z to temperature variations in different parts of the convection roll.)

(a) [2 points] Consider a range of initial conditions taken over some volume in phase space. Demonstrate that as we follow the corresponding trajectories the phase space volume contracts.

(b) [3 points] Find all the fixed points of the Lorenz Equations.

(c) [5 points] Linearize the equations about the fixed point $x = y = z = 0$. Solve your equations to determine the behavior of trajectories near this fixed point for different choices of the parameters.

Interesting aside:

Lorenz (who was an MIT PhD and professor) studied these equations numerically for $\sigma = 10$, $r = 28$, $b = 8/3$. He discovering that the motion is chaotic, and that the system trajectory settles down onto the zero volume "strange attractor" shown here. His work led to a paradigm shift in weather forecasting. He also coined the "butterfly effect" to describe sensitivity to initial conditions.
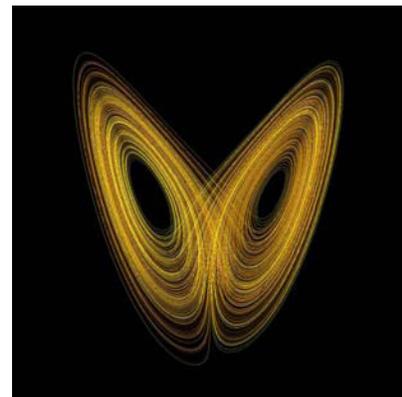
Image is in the public domain.

4. **Chaos in a Nonlinear Circuit** [13 points]

   In lecture we explored trajectories for the damped driven nonlinear oscillator where, with dimensionless variables, the equation of motion was

   $$\ddot{\theta} + \frac{1}{q}\dot{\theta} + \sin\theta = a\cos(w_D t).$$

   To study values of (q, a, w_D) that have chaotic motion we made us of a mathematica demonstration package, where the allowed range for the parameters were $0 < q < 4$, $0 < a < 2$, $-4 < w_D < 4$. You will edit this code to solve the equation below. (Your results should include printouts of the plots you make in mathematica. Note that you can copy & paste the various output panels from the code in order to create a permanent record and enlarge them to get a better printout.)

   As you may recall, in a RLC circuit the current obeys an oscillator equation. In this problem you will explore the chaotic and nonchaotic motion of a RLC circuit with a nonlinear capacitor. For the dimensionless current "x":

   $$\ddot{x} + \frac{1}{q_c}\dot{x} + x^3 = B\cos(w_D t)$$

   Note that parameter values of interest include $0 < q_c < 20$ and $0 < B < 12$.

   Write a suitable set of first order equations for this nonlinear circuit. Modify the mathematica code available on the 8.09 website to implement your nonlinear circuit equations. To keep things simple you should adopt the same notation as the nonlinear driven oscillator and ONLY edit the few lines of the code specifying the equations. The code uses parameters (q, a) with a smaller range than we want for (q_c, B), so define an appropriate mapping $q_c \propto q$ and $B \propto a$.

   (a) [3 points] Take $w_D = 2/3$ and $q_c = 10$. Use the code to create a bifurcation plot showing at least $6 < B < 11$ (choose 300 intervals to get high resolution).

   (b) [6 points] Create Poincaré sections and Phase Portraits showing cases with: period doubling, period quadrupling, and at least two distinct examples with chaos. (Adjust $\Delta\tau$ and the duration to obtain better resolution phase portraits.)

   (c) [4 points] For a chaotic case generate a set of pictures that demonstrate the sensitivity to initial conditions.

5. **Bifurcations** [12 points]

   Consider the equations below with a variable x and real parameter r. For each case determine the critical value of r where a bifurcation occurs, and sketch $\dot{x}$ versus x for any values of r that lead to qualitatively different situations. Also sketch the bifurcation diagram and classify the result as either a saddle-node bifurcation, transcritical bifurcation, or pitchfork bifurcation. If its a pitchfork also say whether its supercritical or subcritical. [Each case is 4 points. Feel free to plot curves with mathematica.]

(a) $\dot{x} = x(r - e^x)$

(b) $\dot{x} = r + x - \ln(1 + x)$

(c) $\dot{x} = x + \tanh(rx)$

8.09 Classical Mechanics III

Fall 2014