# Lecture 10: Oscillators

Harmonic oscillators are ubiquitous in physics. HO behavior appears in any quadratic potential, and most physicists are happy to believe that if you look close enough, all potentials are quadratic.

My aim here and in the next lecture is to explore forced and damped HOs; we will see non-linearities and parametric resonance only numerically, but you can read LL 23-24 for more info.

Let's start from the beginning....

Given a particle at a stable equilibrium q_0 in the potential $U(q)$...
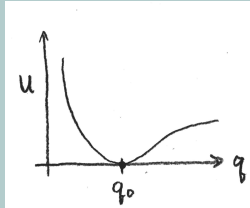
$$U(q) = U(q_0) + \left.\frac{\partial U}{\partial q}\right|_{q_0}(q-q_0) + \frac{1}{2}\left.\frac{\partial^2 U}{\partial q^2}\right|_{q_0}(q-q_0)^2 + \dots$$

take $U(q_0) = 0$ and define $x = q - q_0$

Stable equilibrium $\implies \left.\dfrac{\partial U}{\partial q}\right|_{q_0} = 0$ and $\left.\dfrac{\partial^2 U}{\partial q^2}\right|_{q_0} \geq 0$

$\implies U(x) \approx \frac{1}{2}kx^2$ where $k = \left.\dfrac{\partial^2 U}{\partial q^2}\right|_{q_0}$

So, if the particle stays close enough to q_0, only the quadratic term matters.

If we take x to be a Cartesian coordinate (or even if we don't; see LL page 58), we get the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$
$$\implies m\ddot{x} = -kx \quad \text{or} \quad \ddot{x} + \omega^2 x = 0$$
$$\text{where} \quad \omega^2 = \frac{k}{m}$$

As you saw in 8.01, sines & cosines are solutions to this Diff Eqn:

$$\implies x(t) = a\cos(\omega t + \phi_0)$$
$$\text{since} \quad \ddot{x} = -a\omega^2 \cos(\omega t + \phi_0) = -\omega^2 x$$

The interesting feature of this solution is that it oscillates at a particular frequency, $\omega = \sqrt{\frac{k}{m}}$, which does not depend on the amplitude a, as long as it is **small enough.** (Recall mechanical similarity argument for $U \propto x^2$)

Example. A satellite of mass m (reduced mass μ) in a circular orbit briefly fires a thruster towards the Earth. What is its subsequent motion?
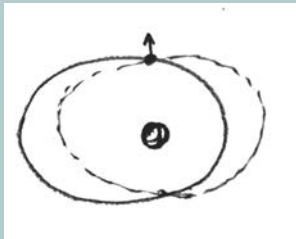
$$U_{\text{eff}}(r) = -\frac{\alpha}{r} + \frac{\lambda}{2r^2} \quad \text{[from lecture 8]}, \quad r_0 = \frac{\lambda}{\alpha}$$

$$\left.\frac{\partial U_{\text{eff}}}{\partial r}\right|_{r_0} = \left.\left(\frac{\alpha}{r^2} - \frac{\lambda}{r^3}\right)\right|_{r_0} = \frac{\alpha^3}{\lambda^2} - \frac{\alpha^3}{\lambda^2} = 0,$$

$$\left.\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right|_{r_0} = \left.\left(-\frac{2\alpha}{r^3} + \frac{3\lambda}{r^4}\right)\right|_{r_0} = -\frac{2\alpha^4}{\lambda^3} + \frac{3\alpha^4}{\lambda^3} = \frac{\alpha^4}{\lambda^3}$$

$$\implies U(x) = \frac{1}{2}kx^2 \quad \text{with } x = r - r_0, \ k = \frac{\alpha^4}{\lambda^3},$$
$$\implies r(t) = r_0 + a\sin(\omega t) \text{ since } r = r_0 \text{ at } t = 0$$
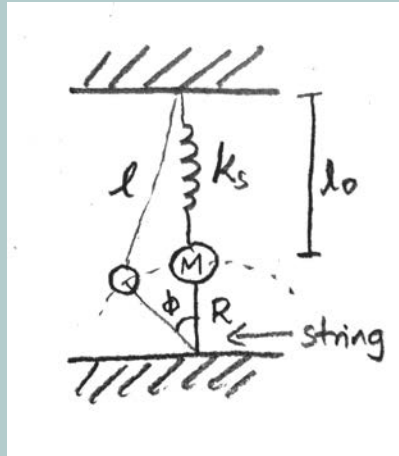
$$\omega = \sqrt{\frac{\alpha^4}{\mu\lambda^3}} \quad \implies \tau_{\text{osc}} = \frac{2\pi}{\omega} = 2\pi\frac{\sqrt{\mu\lambda^3}}{\alpha^2}$$

$$\text{[from Lec 8]} \quad \tau_{\text{orb}} = \pi\alpha\sqrt{\frac{\mu}{-2E^3}}, \quad E_0 = -\frac{\alpha^2}{2\lambda}$$

$$\tau_{\text{orb}} = \pi\alpha\sqrt{\frac{\mu}{2\cdot\alpha^6/(8\lambda^3)}} = 2\pi\frac{\sqrt{\mu\lambda^3}}{\alpha^2}$$
$$\implies \text{closed orbit!!} \quad \text{(as always in } 1/r \text{ potential)}$$

Small perturbation away from circular orbit....

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Example. (using Lagrangian)

$$U = \frac{1}{2}k_s\ell^2 + mgh$$

$$h = R\cos\phi \approx R\left(1 - \frac{1}{2}\phi^2\right)$$

$$\ell^2 = R^2\sin^2\theta + (\ell_0 + R(1-\cos\phi))^2$$
$$\approx \ell_0^2 + R(R+\ell_0)\phi^2$$

$$T = \frac{1}{2}m(R\dot\phi)^2$$

$$\implies L = \frac{1}{2}m(R\dot\phi)^2 - \frac{1}{2}k(R\phi)^2$$

$$\text{for } k = k_s\left(1 + \frac{\ell_0}{r}\right) - \frac{mg}{R}$$

$$\omega^2 = \frac{k}{m} = \frac{k_s}{m}\left(1 + \frac{\ell_0}{R}\right) - \frac{g}{R}$$

note: $k_s\ell_0 = F_s > mg$ or it fails $\implies \omega^2 > 0$

For tomorrow!

- read LL 25-26
- do pset problems 25-27

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