Lecture 11: Forced Oscillations

The previous discussion concerned an HO which, aside from some initial conditions, was free to move without disturbance.

Next, we consider the behavior of an HO with a driving force.

To make a force in Lagrangian Mechanics, we start by modifying the potential:

$$U = \frac{1}{2}kx^2 - xF(t)$$
$$\text{such that } F_x = \frac{\partial L}{\partial x} = -kx + F(t)$$

$$\implies \text{ EoM } m\ddot{x} + kx = F(t)$$
$$or \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \quad \omega_0^2 = \frac{k}{m}$$

Here we enter the land of Diff Eq solving, from whence I know that the solution is the sum of the "general solution" with $F(t) = 0$, and a "particular" solution to account for $F(t)$. We will restrict our attention to oscillatory driving forces, in hopes of finding a workable solution ...

For $F(t) = f\cos(\omega t + \theta)$,

$$x(t) = a_1 \cos(\omega_0 t + \phi) + a_2 \cos(\omega t + \theta)$$

where ϕ, a_1 come from initial conditions, and $a_2 = \dfrac{f}{m(\omega_0^2 - \omega^2)}$

In case you don't trust me...

$$\begin{aligned}
\dot{x} &= -a_1\omega_0 \sin(\omega_0 t + \phi) - a_2\omega \sin(\omega t + \theta) \\
\ddot{x} &= -a_1\omega_0^2 \cos(\omega_0 t + \phi) - a^2\omega^2 \cos(\omega t + \theta)
\end{aligned}$$

$$\begin{aligned}
\ddot{x} + \omega_0^2 x &= \frac{f}{m(\omega_0^2 - \omega^2)} \left(\omega_0^2 - \omega^2\right) \cos(\omega t + \theta) \\
&= \frac{f}{m} \cos(\omega t + \theta) = \frac{F(t)}{m}
\end{aligned}$$

(note: general part cancels, by design; needed to accommodate any possible initial conditions. Particular part matches RHS)

So the motion has 2 parts:

1. A free oscillation, caused by initial conditions.
2. The response to the drive with

$$\begin{aligned}
\frac{x(t)}{F(t)} &= \frac{1}{m(\omega_0^2 - \omega^2)} \\
&\approx \begin{cases} \dfrac{1}{m\omega_0^2} = \dfrac{1}{k} \text{ for } \omega \ll \omega_0 \quad (F = kx \text{ spring w/o mass!}) \\ \text{large for } \omega \approx \omega_0 \quad \text{(resonance)} \\ -\dfrac{1}{m\omega^2} \text{ for } \omega \gg \omega_0 \quad (F = m\ddot{x} \text{ mass w/o spring!}) \end{cases}
\end{aligned}$$

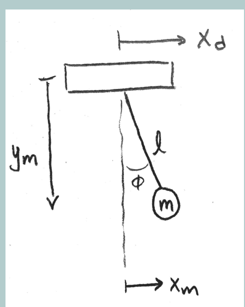
Something bad happens for $\omega = \omega_0$ (drive on resonance). The solution in this case is

$$x(t) = a_1 \cos(\omega_0 t + \phi) + \frac{f}{2m\omega_0} t \sin(\omega_0 t + \theta)$$

in which the response to the driving force increases **linearly** with time, as more and more energy is added to the system.

Clearly this can't go on forever. Either the amplitude gets large and our approximation (or apparatus!) breaks, or some frictional loss stops the growth of the oscillation.

Let's explore this a little with the not-so-simple pendulum.



$$\begin{aligned} U &= -mgy_m \quad = -mg\ell\cos\phi \\ T &= \frac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right) \\ x_m &= x_d + \ell\sin\phi, \quad y_m = \ell\cos\phi \end{aligned}$$

$$\left.\begin{aligned} \dot{x}_m &= \dot{x}_d + \ell\cos\phi\,\dot{\phi} \\ \dot{y}_m &= -\ell\sin\phi\,\dot{\phi} \end{aligned}\right\} \implies T = \frac{1}{2}m\left(\dot{x}_d^2 + \ell^2\dot{\phi}^2 + 2\ell\cos\phi\,\dot{\phi}\,\dot{x}_d\right)$$

$$\begin{aligned} L &= \frac{1}{2}m\left(\ell^2\dot{\phi}^2 + 2\ell\cos\phi\,\dot{\phi}\,\dot{x}_d\right) + mg\ell\cos\phi \\ F_\phi &= \frac{\partial L}{\partial \phi} = -m\ell\sin\phi\,\dot{\phi}\,\dot{x}_d - m\ell g\sin\phi \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m\ell\left(\ell\dot{\phi} + \cos\phi\,\dot{x}_d\right) \end{aligned}$$

For $\phi \ll 1$, we have

$$F_\phi \approx -m\ell(g + \dot{\phi}\dot{x}_d)\phi$$
$$\dot{p}_\phi = m\ell\left(\ell\ddot{\phi} + \ddot{x}_d - \phi\dot{\phi}\dot{x}_d\right)$$

$$\implies \ell\ddot{\phi} + g\phi = -\ddot{x}_d(t)$$
$$\text{for } \phi \ll 1, \quad \ell\phi \approx x_m - x_d \implies \ell\ddot{\phi} = \ddot{x}_m - \ddot{x}_d$$
$$\implies \ddot{x}_m + \frac{g}{\ell}x_m = \frac{g}{\ell}x_d(t).$$

which is our driven HO again with

$$\omega = \sqrt{\frac{g}{\ell}} \qquad F_{d\phi}(t) \quad = -\frac{\ddot{x}_d(t)}{\ell}$$
$$F_{dx}(t) = \frac{g}{\ell}x_d(t)$$

Now, if we remove the assumption that $\phi \ll 1$:

$$\dot{p}_\phi = m\ell\left(\ell\ddot{\phi} + \cos\phi\,\ddot{x}_d - \sin\phi\,\dot{\phi}\,\dot{x}_d\right)$$
$$\implies \ell\ddot{\phi} + g\sin\phi = -\cos\phi\,\ddot{x}_d(t)$$

which we can investigate numerically ...