

— (13) Dissipation and Damped Oscillators —

1 Dissipation

We've been trying to ignore it, but in the real world there is friction. Friction means that mechanical energy is converted to thermal energy, and we no longer have a 'conservative' system. But we can try.

Rather than just start with a damped oscillation (as in eqn 25.1 in LL), I will motivate a modified Euler-Lagrange equation which includes dissipation, and then use this to arrive at damped oscillations.

Imagine some fraction of kinetic energy is couple to thermal energy per unit time ϵ .

$$L'(q, \dot{q}) = (T + T_\mu) - U = L + \int \epsilon T dt$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}} \right) &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{d}{dt} \left(\int \frac{\partial}{\partial \dot{q}} (\epsilon T) dt \right) \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial}{\partial \dot{q}} (\epsilon T) = \frac{\partial L'}{\partial \dot{q}} \\ &\quad \text{given uniform } \epsilon \text{ such that } \frac{\partial L'}{\partial q} = \frac{\partial L}{\partial q} \\ \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) &= \frac{\partial L}{\partial q} - \frac{\partial}{\partial \dot{q}} (\epsilon T) \end{aligned}$$

Generalizing ϵT to any velocity dependent function,

$$\text{Let } D = \frac{1}{2} \sum_{j,k} b_{jk} \dot{q}_j \dot{q}_k \quad \text{“dissipative function”}$$

or, for one degree of freedom, simply

$$D = \frac{1}{2} b \dot{q}^2$$

modified E-L

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} - \frac{\partial D}{\partial \dot{q}}$$

such that $\dot{p} = \left(\frac{\partial L}{\partial q} - b\dot{q} \right)$

where $\frac{\partial L}{\partial q}$ is the conservative force, and $b\dot{q}$ is the dissipative force.

Damped systems lose energy with time until they come to rest. The rate of energy loss is given by the dissipation function.

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \left(\dot{q} \frac{\partial L}{\partial \dot{q}} - L \right) \\ &= \dot{q} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial \dot{q}} \ddot{q} - \left(\frac{\partial L}{\partial t} + \dot{q} \frac{\partial L}{\partial q} + \ddot{q} \frac{\partial L}{\partial \dot{q}} \right) \\ &= \dot{q} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right) \quad \text{normally zero, but...} \\ \frac{dE}{dt} &= -\dot{q} \frac{\partial D}{\partial \dot{q}} = -2D \end{aligned}$$

Note that the last line is just the rate of work done by friction as force \times velocity. All of this is assuming no external driving force (i.e., $\frac{\partial L}{\partial t} = 0$).

2 Damped Oscillator

Let's put this to work on our harmonic oscillator to make a more realistic damped oscillator.

$$\begin{aligned} \text{for } L &= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \\ \text{and } D &= \frac{1}{2} b \dot{x}^2 \end{aligned}$$

the equations of motion are

$$m\ddot{x} = -kx - b\dot{x}, \quad \omega_0^2 = \frac{k}{m}$$

or

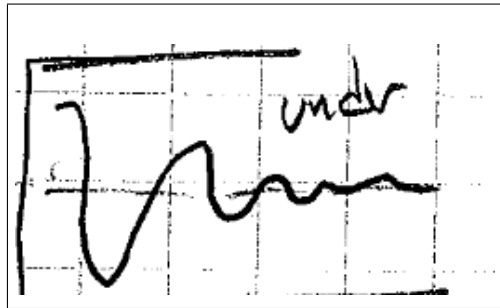
$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2x = 0, \quad 2\lambda = \frac{b}{m}$$

This differential equation is best solved with complex exponentials, but the solution can be written in real form as

“under damped”

$$x(t) = ae^{-\lambda t} \cos(\omega_1 t + \phi) \text{ for } \lambda < \omega_0$$

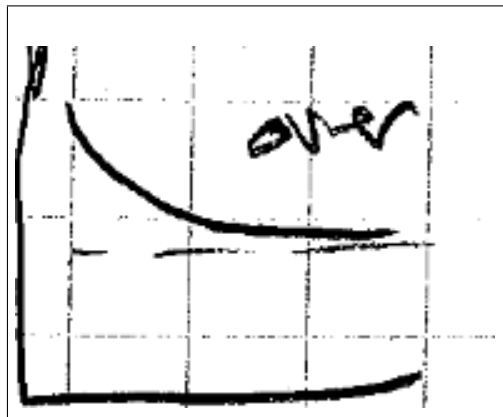
where $\omega_1 = \sqrt{\omega_0^2 - \lambda^2}$



“over damped”

$$x(t) = e^{-\lambda t} (a_1 e^{\beta t} + a_2 e^{-\beta t}) \text{ for } \lambda < \omega_0$$

where $\beta = \sqrt{\lambda^2 - \omega_0^2}$, note $\beta < \lambda \Rightarrow$ decay



“critically damped”

$$x(t) = e^{-\lambda t} (a_1 + a_2 t) \quad \text{for } \lambda = \omega_0$$

To complete the picture, we should add a driving force to our damped oscillator. Returning to the equation of motion...

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{F(t)}{m} = \frac{f}{m} \cos(\omega t)$$

$$\Rightarrow x(t) = a_1 e^{-\lambda t} \cos(\omega_0 t + \phi) + a_2 \cos(\omega t + \theta)$$

$$a_2 = \frac{f}{m \sqrt{(\omega^2 - \omega_0^2)^2 + 4\lambda^2 \omega^2}}$$

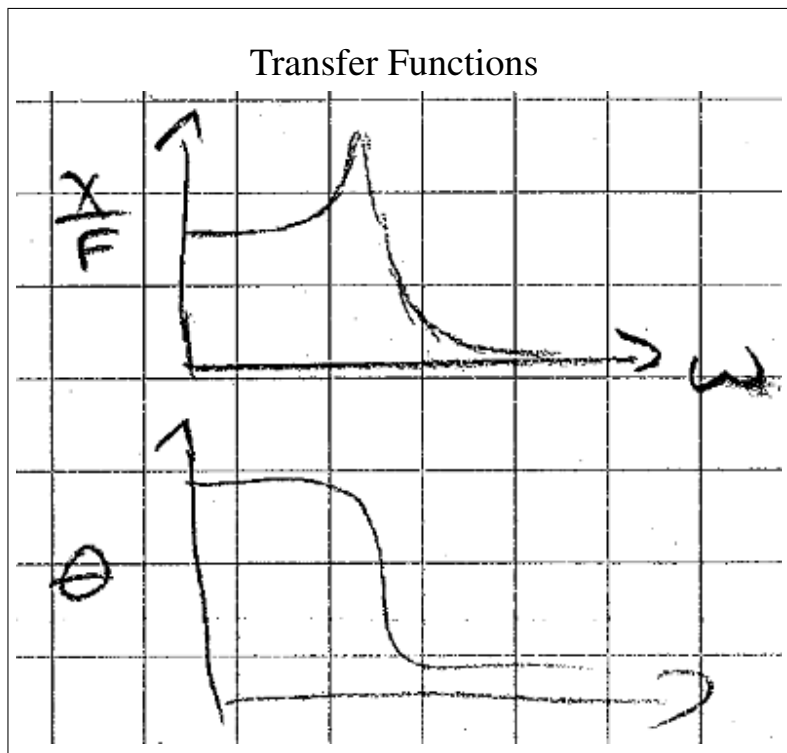
$$\tan \theta = \frac{2\lambda\omega}{\omega^2 - \omega_0^2}$$

where a_1 and ϕ come from the initial conditions

Again, the driven solution has 2 parts, one that depends on the initial conditions and another which is the response to the drive. With damping, we see that the first of these decays with time, such that the motion at $t \gg \frac{1}{\lambda}$ is essentially only the driven response.

$$x(t) \simeq a_2 \cos(\omega t + \theta) \quad \text{for } t \gg \frac{1}{\lambda} \quad (1)$$

$$\rightarrow \frac{x(t)}{F(t)} = \frac{a_2}{f} \quad (2)$$



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