Lecture (4)

Last time: Mechanics of Lagrangian Mechanics
Today: Symmetries and Galilean Relativity

<u>for tomorrow</u>

1. read LL 6-8
2. finish pset 1 by class time!

1 Nature of Lagrangian and PLA

Let's take a step back and note some interesting features of Lagrangian mechanics.

L is a scalar function (not a vector like $\vec{F}$)
$L' = \alpha L$ or $L'' = L + \alpha$ give the same EoM for any constant α (PLA)

The fact that scaling and offsetting L have no effect is clear from the PLA: the true path is unchanged. Note also that L is additive:

$L_{ab} = L_a + L_b = (T_a + T_b) - (U_a + U_b)$
for independent particles a and b

for interacting particles, we use a potential which depends on both particles

Interacting Particles

$$L_{ab} = T_a(q_a, \dot{q}_a) + T_b(q_b, \dot{q}_b) - U_{ab}(q_a, q_b)$$

more on interactions later.

In addition to constants, we can add any *total time derivative* to L without changing the resulting physics (EoMs). See LL eq 2.8.

$$
\begin{aligned}
L' &= L + \frac{\mathrm{d}}{\mathrm{d}t} f(q,t) \\
S' &= \int_{t_1}^{t_2} L + \frac{\mathrm{d}}{\mathrm{d}t} f(q,t) dt = S + f(q,t)|_{t_1}^{t_2} = S + \alpha
\end{aligned}
$$

Since all trial paths pass through the same endpoints at t_1 and t_2, $f(q,t)|_{t_1}^{t_2}$ is a *constant*.

For the curious, I will say that this $f(t)$ is the same as the "generator functions" which will appear briefly at the end of this class (eqn 45.7), but its deep implications are revealed only in 8.09.

For now, let's look at some examples.

$\Rightarrow$ Ask class for examples! $\Leftarrow$

$$
\begin{aligned}
g(q,\dot{q},t) &= \frac{\mathrm{d}}{\mathrm{d}t} f(q,t) \\
g(q,\dot{q},t) &= h(t) \Rightarrow f(t) = \int h(t) dt \text{ for any } h(t) \\
\text{or} \quad & \alpha \rightarrow f(t) = \alpha t \text{ for any constant } \alpha \\
\text{or} \quad & h(q)\dot{q} \rightarrow \text{ exists } f \text{ such that } h(q) = \frac{\partial f(q)}{\partial q} \\
& \quad \rightarrow \frac{d}{dt} f(q) = \frac{\partial f(q)}{\partial t} + \frac{\partial f(q)}{\partial q} \dot{q} = h(q)\dot{q}
\end{aligned}
$$

You can also use this to simplify life:

$$
\begin{aligned}
g(q,\dot{q},t) &= a(q,t) \frac{\mathrm{d}}{\mathrm{d}t} b(q,t) \\
&= \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(a\,b)}_{\text{drop}} - b \frac{\mathrm{d}}{\mathrm{d}t} a
\end{aligned}
$$

This type of trick is used in the text with little warning or explanation!

2 Symmetries of Space and Time

In order to give concrete examples, I have already given you some idea of the fundamental form of the Lagrangian.

$L = T - U = \frac{1}{2}mv^2 - U(q)$
for a free particle $U(q) = 0$

Now I will prove that L must be like this using only the PLA and symmetry. I will use Cartesian coordinates for all of this (though I will still write q) because other coordinates are more complicated and not more enlightening. We start with the simplest of assumptions...

Free particle, all points in space are equal
(space is homogeneous)

$$\Rightarrow L(\vec{q}, \dot{\vec{q}}, t) \to L(\dot{\vec{q}}, t)$$

can't depend on WHERE

all points in time are equal (time is homogeneous)

$$\Rightarrow L(\dot{\vec{q}}, t) \to L(\dot{\vec{q}})$$

can't depend on WHEN

Euler-Lagrange

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} = \frac{\partial L(\dot{q})}{\partial q} = 0$$

$$\Rightarrow \frac{\partial L(\dot{q})}{\partial \dot{q}} = \text{ const } \Rightarrow L = \alpha\dot{q}, \text{ or } \dot{q} = \text{ const}$$

all directions are equal (space is isotropic)

$$\Rightarrow L\left(\dot{\vec{q}}\right) \to L\left(|\dot{\vec{q}}|\right) \to L\left(\dot{\vec{q}} \cdot \dot{\vec{q}}\right) = L\left(v^2\right)$$

$$v^2 = \sum_i \dot{q_i}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

So L cannot be proportional to $\dot{\vec{q}}$, or else direction would matter. Simply put

$$\text{if } L(\dot{q}) = \alpha\dot{q} \ \text{ then } L(\dot{q}) \neq L(-\dot{q}) = -\alpha\dot{q}$$
$$\text{if } L(\dot{q}) = \alpha\dot{q}^2 \ \text{ then } L(\dot{q}) = L(-\dot{q})$$

Thus we know that the Lagrangian of a free particle can only depend on the magnitude of its velocity, and we know that the result will be motion with constant velocity (vectorially).

Newton's 1st Law, but with nothing more than PLA in homogeneous and isotropic space and time.

Let's try a simple example to see how this works.

$$\begin{aligned} L &= a + bv + cv^2 + dv^3 + \ldots \qquad \text{where } v = \dot{q} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(b + 2cv + 3dv^2 + \ldots) = 0 \\ 0 &= 2c\dot{v} + 3d(2v\dot{v}) + \ldots = (2c + 6dv + \ldots)\dot{v} \end{aligned}$$

So the 1st and 2nd terms don't survive the differentiation, and if $c \neq 0$, we must have $\dot{v} = 0 \Rightarrow v = \text{const}$. More terms don't change this, and actually all odd order terms are inconsistent with isotropic space ($L(\dot{q}) = L(-\dot{q})$).

NB: we picked Cartesian coords because the unit vectors are the same everywhere and interchangeable under rotation (e.g. homogeneous and isotropic).

For a free particle in Cart' coords, we can only have $L = \sum_{n=1} \alpha_n \dot{q}^{2n}$

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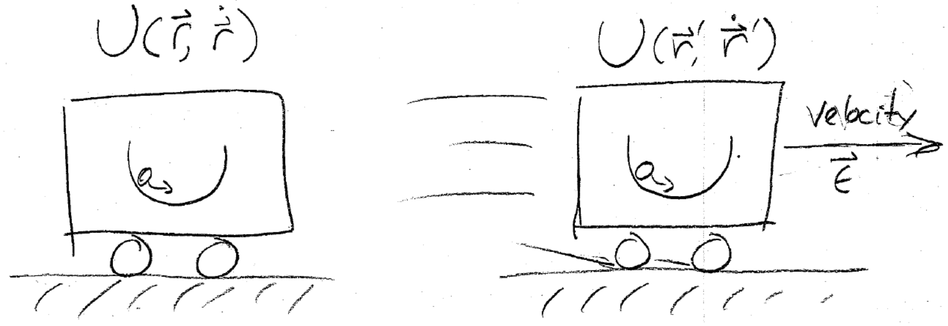
3 Galilean Relativity

What can we say using Galilean Relativity?

We'll start with 2 train cars, moving at some relative velocity $\vec{\epsilon}$.

Galilean Box Cars



$$\begin{aligned} U(\vec{r}, \dot{\vec{r}}) &= \text{potential which defines the experiment in each car} \\ U(\vec{r}\,', \dot{\vec{r}}\,') &= U(\vec{r}, \dot{\vec{r}}) \quad \text{same in local coordinates} \\ \vec{r}\,' &= \vec{r} + \vec{\epsilon}t\ , \ \ \dot{\vec{r}}\,' = \dot{\vec{r}} + \vec{\epsilon} \quad \text{relative motion} \end{aligned}$$

the Lagrangians for each car are

$$\begin{aligned} L = L(\vec{r}, \dot{\vec{r}}) &= L_{FP} - U(\vec{r}, \dot{\vec{r}}) \\ L' = L'(\vec{r}\,', \dot{\vec{r}}\,') &= L_{FP} - U(\vec{r}\,', \dot{\vec{r}}\,') \\ \text{where } L_{FP} &= L_{FP}(v^2) \quad \text{Lagrangian of free particle} \\ \text{and} \quad v^2 &= |\dot{\vec{r}}|^2 \end{aligned}$$

Galilean Relativity tells us that the physics in the two cars is the same. This means that the Equations of Motion must be the same. We also know that the PLA tells us that 2 Lagrangians give the same physics if

$$L' = L + \frac{\mathrm{d}}{\mathrm{d}t} f(q,t) \Rightarrow \text{ same physics}$$

so...

$$
\begin{aligned}
L - L' &= L_{FP}(v^2) - L_{FP}(v'^2) - \Big(U(\vec{r}, \dot{\vec{r}}) - U(\vec{r}\,', \dot{\vec{r}}\,')\Big) \\
&= L_{FP}(v^2) - L_{FP}(v'^2)
\end{aligned}
$$

Earlier we expanded $L_{FP}(v)$ in powers of v and found that L_{FP} could only contain even powers v, leaving us with L_{FP} as some function of v^2. Since Galilean Relativity holds for small velocities as well as large ones, we can assume that ϵ is small and expand $L_{FP}(v^2)$ around $\epsilon = 0$.

Expand $L_{FP}(v'^2)$ around $\epsilon = 0$

$$
\begin{aligned}
L_{FP}(v'^2) &= L_{FP}(v^2 + 2\,\dot{\vec{r}} \cdot \vec{\epsilon} + \epsilon^2) \\
&\simeq L_{FP}(v^2) + \frac{\partial L_{FP}}{\partial (v^2)} \Big(2\,\dot{\vec{r}} \cdot \vec{\epsilon} + \epsilon^2\Big) \\
\Rightarrow L' - L &\simeq \frac{\partial L_{FP}}{\partial (v^2)} \Big(2\,\dot{\vec{r}} \cdot \vec{\epsilon} + \epsilon^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L_{FP}}{\partial (v^2)} \Big(2\dot{\vec{r}} \cdot \vec{\epsilon} + \epsilon^2\Big) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L_{FP}}{\partial (v^2)} \left(2\vec{r} \cdot \vec{\epsilon} + \epsilon^2 t\right) \right) - \\
&\qquad \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_{FP}}{\partial (v^2)} \right) \left(2\vec{r} \cdot \vec{\epsilon} + \epsilon^2 t\right)
\end{aligned}
$$

In order for us to be sure that the two Lagrangians to give the same Equations of Motion, we need the second term to be zero for all $\vec{r}$ and any small ϵ. That would make $L' - L$ equal to the total time derivative of some function $f(q, t)$.

Same Physics

$$
\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_{FP}}{\partial (v^2)} = 0 \Rightarrow \frac{\partial L_{FP}}{\partial v^2} = \text{ constant } \Rightarrow L_{FP} = K v^2
$$

Thus, $L = Kv^2$ without higher powers of v^2 is consistent with Galilean Relativity. We choose $K \propto m$ to maintain the additive nature of L, and $K = \frac{1}{2}m$ to make $F = ma$ pretty.

4 Newton's Laws

Let's finish deriving Newton's Laws from the PLA and basic symmetries. I have already shown that we can get $F = ma$ from E-L, but previously I always *assumed* that $L = T - U$. Now we know that in Cartesian coordinates, with homogeneous space and time, isotropic space, and requiring Galilean Relativity,

$$\begin{aligned} L &= \frac{1}{2}mv^2 - U = T - U \\ \frac{\partial L}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} &\Rightarrow F = ma \end{aligned}$$

So we have now arrived at $F = ma$ from PLA and broad assumptions about space and time! We are also in a position to consider interacting particles

Interacting Particles

$$\begin{aligned} L &= \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2 - U(\vec{r}_a - \vec{r}_b) \\ \text{for } x_a : \frac{\partial L}{\partial x_a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_a} &= -\frac{\partial U}{\partial x_a} - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_a) \\ &= F_{x_a} - m\ddot{x}_a = 0 \end{aligned}$$

Extending this to all coordinates we find Newton's 3rd Law:

$$-\nabla_a U = m_a\ddot{\vec{r}}_a \text{ and } -\nabla_b U = m_b\ddot{\vec{r}}_b$$
$$U(\vec{r}_a - \vec{r}_b) \Rightarrow m_a\ddot{\vec{r}}_a = \vec{F}_{ab} = -m_b\ddot{\vec{r}}_b$$

"For every action, there is an equal and opposite reaction."

Questions on this?

5 Math Aside

Again, before we move on to more physics, let's look for a moment at these tricky derivatives.

On the pset you were asked to show that

$$\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}f(q,\dot{q},t) \neq \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}f(q,\dot{q},t)$$

$$\text{even though } \frac{\partial}{\partial q}\frac{\mathrm{d}}{\mathrm{d}t}f(q,\dot{q},t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial q}f(q,\dot{q},t)$$

Let's explore this a little bit...

$$\begin{aligned}
\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}f &= \frac{\partial}{\partial \dot{q}}\left(\frac{\partial}{\partial t}f + \dot{q}\frac{\partial}{\partial q}f + \ddot{q}\frac{\partial}{\partial \dot{q}}f\right) \\
&= \frac{\partial}{\partial t}\frac{\partial}{\partial \dot{q}}f + \frac{\partial}{\partial q}f + \dot{q}\frac{\partial}{\partial q}\frac{\partial}{\partial \dot{q}}f + \left(\frac{\partial}{\partial \dot{q}}\ddot{q}\right)\frac{\partial}{\partial \dot{q}}f + \ddot{q}\frac{\partial}{\partial \dot{q}}\frac{\partial}{\partial \dot{q}}f
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}\dot{q} &\text{ looks like } \frac{\partial}{\partial x}\frac{\mathrm{d}}{\mathrm{d}t}g(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{1} = 0
\end{aligned}$$

All but one of the remaining terms can be grouped to make $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}f$ such that

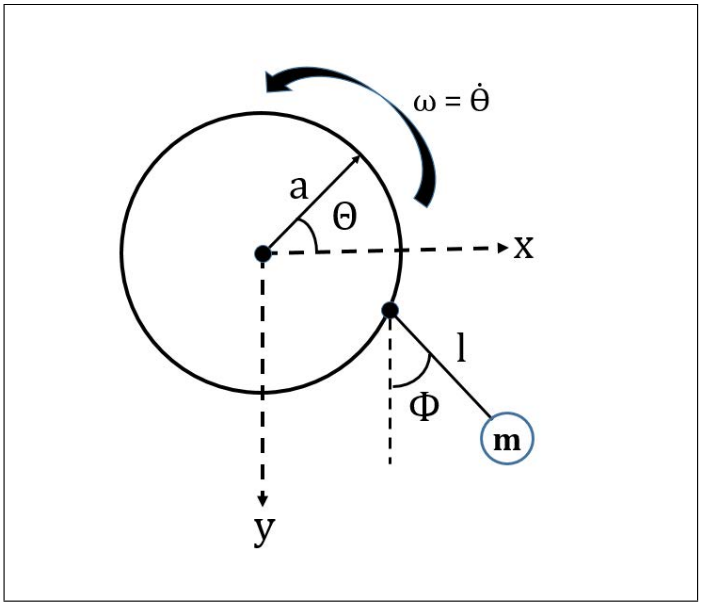
$$\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}f = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}f + \frac{\partial}{\partial q}f$$

$$\Rightarrow \frac{\partial}{\partial \dot{q}} \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} \text{ do not commute}$$

NB: $\frac{\partial}{\partial \dot{q}} = \frac{\partial}{\partial t}(\frac{\partial \dot{q}}{\partial t})^{-1}$

6 Example

Let's apply all that we have learned to a moderately complicated problem (5.3 in LL) to find the EoMs. Here we have a mass on a string attached to a rotating wheel. The wheel rotates at a fixed angular velocity ω, and gravity acts on the mass m.



$$
\begin{aligned}
x &= a\cos(\omega t) + l\sin\phi \\
y &= -a\sin(\omega t) + l\cos\phi \\
\text{find } L &= T - U \\
U &= -mgy \ \text{ and } \ T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\
\dot{x} &= -a\omega\sin(\omega t) + l\cos\phi\dot{\phi} \\
\dot{y} &= -a\omega\cos(\omega t) - l\sin\phi\dot{\phi} \\
\Rightarrow T &= \frac{1}{2}m(a^2\omega^2\sin^2(\omega t) - 2a\omega l\sin(\omega t)\cos\phi\,\dot{\phi} + l^2\cos^2\phi\,\dot{\phi}^2 \\
&\quad + \underbrace{a^2\omega^2\cos^2(\omega t)}_{a^2\omega^2 \to \ drop!} + \underbrace{2a\omega l\cos(\omega t)\sin\phi\,\dot{\phi}}_{2a\omega l\sin(\phi - \omega t)\dot{\phi}} + \underbrace{l^2\sin^2\phi\,\dot{\phi}^2}_{l^2\dot{\phi}^2}
\end{aligned}
$$

Note:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cos(\phi - \omega t)) = \sin(\phi - \omega t)(\omega - \dot{\phi})$$

$$\Rightarrow \sin(\phi - \omega t)\dot{\phi} = \omega \sin(\phi - \omega t) - \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(\ldots)}_{drop!}$$

$$\Rightarrow T = \frac{1}{2}m(2al\omega^2 \sin(\phi - \omega t) + l^2\dot{\phi}^2)$$

$$U = -mgy = mg(\underbrace{a\sin(\omega t)}_{f(t)\to drop!} - l\cos\phi)$$

This finally brings us to the Lagrangian given in LL problem 5.3a (they give no explanation!)

$$L = ml(a\omega^2 \sin(\phi - \omega t) + \frac{1}{2}l\dot{\phi}^2 + g\cos\phi) \text{ [note units]}$$

$$\Rightarrow F_\phi = \frac{\partial L}{\partial \phi} = ml(a\omega^2 \cos(\phi - \omega t) - g\sin\phi) \text{ [torque]}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} \; [I\omega \Rightarrow \text{ angular momentum}]$$

$$F_\phi = \dot{p}_\phi \Rightarrow \ddot{\phi} = \frac{a}{l}\omega^2 \cos(\phi - \omega t) - \frac{g}{l}\sin\phi$$

Where in the last step I have made the E-L equation look like Newton's 2nd Law.

for tomorrow

1. *read LL 6-8*
2. *finish pset 1 by class time!*