Lecture 5 - Topics

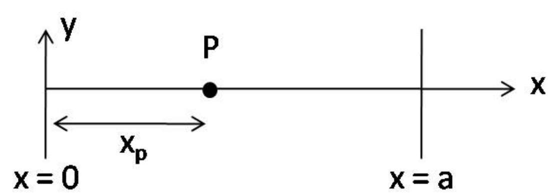
- Nonrelativistic strings
- Lagrangian mechanics

Reading: Zwiebach, Chapter 4

Non-Relativistic Strings

Study nonrelativistic strings first to develop intuition and math notation before moving to the relativistic strings that we actually care about.

Non-relativistic string:



Characterized by:
Tension, T_0: $[T_0]$ = [Force] = [Energy/Length] = $\frac{M}{L}$ $[v^2]$
Mass/Length: μ_0
$T_0 \approx \mu_0 v^2$
Natural velocity: $v = \sqrt{T_0/\mu_0}$

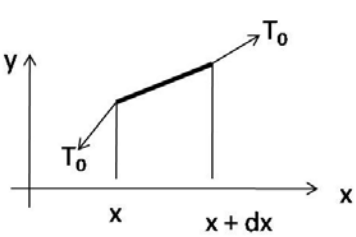
Transverse Oscillation: Mark point P on string and see it moving up and down:

$$y(P,t), x(P,t) = x(P) \qquad (x \text{ not dependent on } t)$$

Small Oscillation:

$$\left|\frac{\partial y}{\partial x}(t,x)\right| << 1$$

Consider small section of string:



Approximate tensions on endpoints as equal (good for transverse waves, terrible for longitudinal)

$$\begin{aligned} dF_\nu &= T_0 \frac{\partial y}{\partial x}(t, x+dx) - T_0 \frac{\partial y}{\partial x}(t,x) \\ &= T_0 \frac{\partial^2 y}{\partial x^2}(t,x)dx \\ &\approx \mu_0 dx \frac{\partial^2 y}{\partial t^2} \end{aligned}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{T_0/\mu_0}\frac{\partial^2 y}{\partial t^2} = 0}$$

The Wave Equation! t, x are parameters. Motion described by $y(t,x)$. (If had motion in more than 1 dimension $\vec{y}(t,x)$)

Stretching of string:

$$\begin{aligned} \Delta l &= \sqrt{dx^2 + dy^2} - dx \\ &= dx(\sqrt{1 + (dy/dx)^2} - 1) \\ &= \frac{1}{2} dx (dy/dx)^2 \qquad ((small)) \end{aligned}$$

General form of wave equation:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

v: velocity of wave, $v = \sqrt{T_0/\mu_0}$

General Solution:

$$y(x,t) = h_+(x - v_0 t) + h_-(x + v_0 t)$$

Note: the h's are function of 1 variable $(x \pm v_0 t)$ not 2 variables x and t independently.

Boundary Conditions: Behavior of endpoints at all times (special points at all times)

Open string:

$$y(t, x=0) = 0 \qquad \text{(Dirichlet condition - for fixed end point)}$$

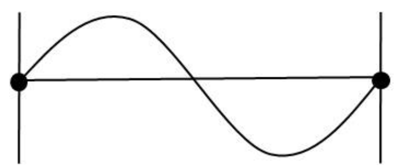
$$\frac{\partial y}{\partial x}(t, x=0) = 0 \quad \text{(Free BD, Neumann condition)}$$

For free endpoint (hoop on string), means string must be perp. here

Initial Conditions: All points on string at some t_0 (all points at special time)

$$y(\lambda, t = 0)$$
$$\frac{\partial y}{\partial x}(x, t = 0)$$

Example:



Fixed Endpoints:

$$\begin{aligned} y(t, 0) &= h_+(-v_0 t) + h_-(v_0 t) = 0 \qquad \text{Let } u = v_0 t \\ &= h_+(-u) + h_-(u) \end{aligned}$$

$$\boxed{h_-(u) = -h_+(-u)}$$

$$y(t, x = a) = 0 = h_+(a - v_0 t) + h_-(a + v_0 t)$$
$$h_+(a - v_0 t) = -h_-(a + v_0 t) = h_+(-a - v_0 t)$$

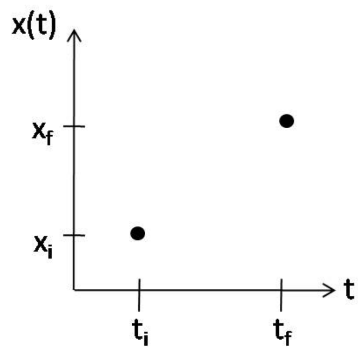
Let $u = -a - v_0 t$

$$\boxed{h_+(u + 2a) = h_+(u)}$$

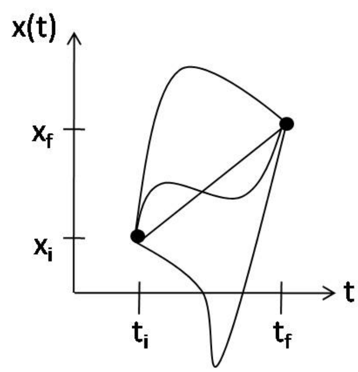
Variational Principle

Consider point mass m doing 1D motion $x(t)$.
Assume $x(t_i) = x_i$, $x(t_f) = x_f$. Under the influence of potential $V(x)$

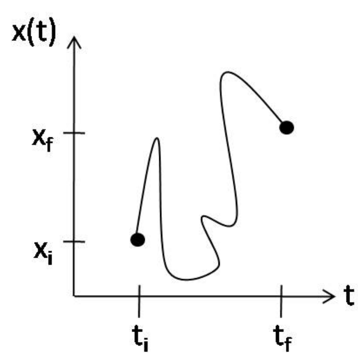
Know:



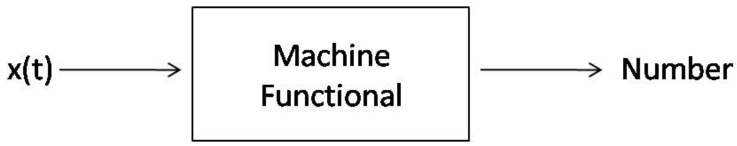
Possible motions:



Not possible:



Given a path:



Functional: $S : x(t) \Rightarrow \Re$ (not a function of time)

Hamilton's Principle: Principal path makes S stationary.

Call true path $x(t)$. Consider new path $x(t) + \delta x(t)$

$$S[x(t) + \delta x(t)] = S[x] + \theta[(\delta x)^2]$$

Assume $\delta x(t_i) = 0$, $\delta x(t_f) = 0$

Lagrangian:

$$L(t) = \text{Kinetic Energy - Potential Energy}$$

$$S = \int_{t_1}^{t_2} L(t)dt = \int_{t_1}^{t_2} \left[\frac{1}{2}m(\dot{x}(t))^2 - V(x(t))\right]dt$$

$$\begin{aligned} S[x + \delta x] &= \int_{t_i}^{t_f} \left[\frac{1}{2}m(\dot{x} + \delta\dot{x})^2 - V(x + \delta x)\right]dt \\ &= S[x] + \int_{t_i}^{t_f} \left[m\dot{x}\delta\dot{x} - \frac{\partial V}{\partial x}(x(t)\delta x(t))\right]dt + \underbrace{\int_{t_i}^{t_f} \frac{1}{2}m(\delta\dot{x}(t))^2 - \frac{1}{2}V''(\delta x)^2}_{\theta(\delta x^2)} \end{aligned}$$

Need to eliminate second term.

$\int_{t_i}^{t_f} [m\dot{x}\delta\dot{x} - \frac{\partial V}{\partial x}(x(t)\delta(x(t)))]dt$ must go away for $S[x + \delta x] = S[x] + \theta[(\delta x)^2]$ to be true.

Call this the variation δS

$$\delta S = \int_{t_i}^{t_f} dt\left[\frac{d}{dt}(m\dot{x}\delta x) - m\ddot{x}\delta x - V'(x(t))\delta(x(t))\right]$$

Integrate by parts

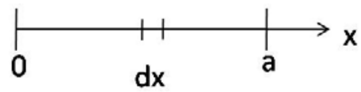
$$dS = m\dot{x}(t_f)\delta x(t_f) - m\dot{x}(t_i)\delta x(t_i) + \int_{t_i}^{t_f} dt\delta x(t)[-m\ddot{x} - V'(x(t))]$$

$\delta x(t_f) = \delta(t_i) = 0$ from before.

The integral $\int_{t_i}^{t_f} dt\delta x(t)[-m\ddot{x} - V'(x(t))]$ must be 0 too, so:

$$m\ddot{x} = -V'(x(t))$$

String Lagrangian



T: Kinetic energy $= \frac{1}{2}\mu_0 dx\left(\frac{\partial y}{\partial t}\right)^2$

Potential Energy $= \sum_{\text{string}} \Delta l T_0 = \int_0^a \frac{1}{2}dx\left(\frac{\partial y}{\partial x}\right)^2 T_0$

$$L = \int_0^a dx\left[\frac{1}{2}\mu_0(\partial y/\partial t)^2 - \frac{1}{2}T_0(\partial y/\partial t)^2\right]$$

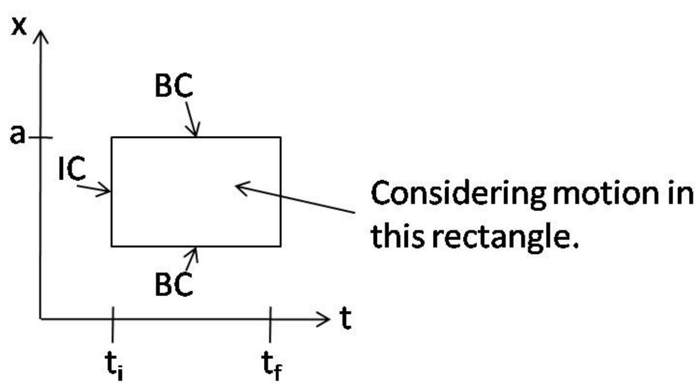
$$S = \int_{t_i}^{t_f} L(t)dt$$

Call $\mathcal{L}$: Lagrangian Density

$$\mathcal{L} = \frac{1}{2}\mu_0(\frac{\partial y}{\partial t})^2 - \frac{1}{2}(\frac{\partial y}{\partial t})$$

So:

$$S = \int_{t_i}^{t_f} dt \int_0^a dx\mathcal{L}\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right)$$



$$\delta y(t_i, x) = 0$$
$$\delta y(t_f, x) = 0$$

Don't know $\delta y(x = 0, t)$ or $\delta y(x = a, t)$

$$\delta S = \int_{t_i}^{t_f} dt \int_0^a dx\left[\frac{\partial \mathcal{L}}{\partial \dot{y}}\delta\dot{y} + \frac{\partial \mathcal{L}}{\partial y'}\delta y'\right]$$

Let:

$$\mathcal{P}^t = \partial\mathcal{L}/\partial\dot{y}$$
$$\mathcal{P}^x = \partial\mathcal{L}/\partial y'$$

$$\delta S = \int_{t_i}^{t_f}\int_0^a \left[\mathcal{P}^t \frac{\partial(\delta y)}{\partial t} + \mathcal{P}^x \frac{\partial(\delta y)}{\partial x}\right]$$

$$\delta S = \int_{t_i}^{t_f} dt \int_0^a dx \left[-\delta y(x,t)\left(\frac{\partial\mathcal{P}^t}{\partial t} + \frac{\partial\mathcal{P}^x}{\partial x}\right)\right] + \int_0^a dx \mathcal{P}^t [\delta y]_{t_i}^{t_f} + \int_{t_i}^{t_f} \mathcal{P}^x [\delta y]_{x=0}^{x=a}$$

$\delta y(t_i) = \delta y(t_f) = 0$

Must have:

$$\frac{\partial\mathcal{P}^t}{\partial t} + \frac{\partial\mathcal{P}^x}{\partial x} = 0 = \mu_0 \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial x^2}$$

Some kind of conservation law like $\partial_\mu J^\mu = 0$

$$\int_{t_i}^{t_f} dt \mathcal{P}^x [\delta y]_{x=0}^{x=a} = \int_{t_i}^{t_f} dt[\mathcal{P}^x(t, x=a)\delta y(t, x=a) - \mathcal{P}^x(t, x=0)\delta y(t, x=0)]$$

For $* \in 0, a$:

$$\mathcal{P}^x(t, x_*)\delta y(t, x_*)$$

Dirichlet condition:
$y(t, x_*) =$ fixed, $\delta y(t, x_*) = 0$
Free boundary condition:
$\mathcal{P}^x(t, x_*) = 0$, $\partial y/\partial x = 0$ (Neumann condition)