MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

Earth, Atmospheric, and Planetary Sciences Department

Astronomy 8.282J-12.402J March 13, 2006

Quiz 1

Name Solutions **(please print)**

Last **First**

1. Be sure to attempt all problems.
2. The point values of the problems are indicated at the top of each page.
3. Closed book exam; you may use one page of notes.
4. Wherever possible, try to solve the problems using general analytic expressions. Plug in numbers only as a last step.

Problem	Grade	Grader
1		
2		
3		
4		
Total		

APPROXIMATE VALUES OF USEFUL CONSTANTS

Constant	cgs	units	mks	units
c (speed of light)	3×10^{10}	cm/sec	3×10^{8}	m/sec
G (gravitation constant)	7×10^{-8}	dyne-cm^2/g^2	7×10^{-11}	N-m^2/kg^2
k (Boltzmann's constant)	1.4×10^{-16}	erg/K	1.4×10^{-23}	J/K
h (Planck's constant)	6.6×10^{-27}	erg-sec	6.6×10^{-34}	J-sec
$m_{\rm proton}$	1.6×10^{-24}	g	1.6×10^{-27}	kg
eV (electron Volt)	1.6×10^{-12}	erg	1.6×10^{-19}	J
$M_\odot$ (solar mass)	2×10^{33}	g	2×10^{30}	kg
$L_\odot$ (solar luminosity)	4×10^{33}	erg/sec	4×10^{26}	J/sec
$R_\odot$ (solar radius)	7×10^{10}	cm	7×10^{8}	m
σ (Stefan-Boltzmann cons)	6×10^{-5}	erg/cm^2-sec-K^4	6×10^{-8}	J/m^2-sec-K^4
Å(Angstrom)	10^{-8}	cm	10^{-10}	m
km (kilometer)	10^{5}	cm	10^{3}	m
pc (parsec)	3×10^{18}	cm	3×10^{16}	m
kpc (kiloparsec)	3×10^{21}	cm	3×10^{19}	m
Mpc (megaparsec)	3×10^{24}	cm	3×10^{22}	m
year	3×10^{7}	sec	3×10^{7}	sec
day	86400	sec	86400	sec
AU	1.5×10^{13}	cm	1.5×10^{11}	m
$1'$ (arc minute)	1/3400	rad	1/3400	rad
$1''$ (arc second)	$1/200,000$	rad	$1/200,000$	rad

Problem 1 (30 points)

Short Problems in Gravity

(a) Find the gravitational acceleration due to the Sun at the location of the Earth's orbit (i.e., at a distance of 1 AU).

$$\begin{aligned}
F &= ma = \frac{GM_\odot m}{r^2} \\
a &= \frac{GM_\odot}{r^2} \\
a &= \frac{7\times10^{-11}\text{N-m}^2/\text{kg}^2 \cdot 2\times10^{30}\text{kg}}{(1.5\times10^{11}\text{m})^2} \\
a &= 0.006\text{m/s}^2
\end{aligned}$$

(b) Show that for a small mass, m, orbiting a much larger mass, M, in an eccentric orbit, that the area swept out per unit time by a line joining the large and small mass is a constant around the orbit (i.e., Kepler's 2nd law). Hint: use the appropriate conservation law.

$$dA \simeq \frac{1}{2}r^2 d\theta + \frac{1}{2} r\, d\theta\, dr$$

We note that the $\frac{1}{2}rd\theta dr$ term is negligible.

$$\begin{aligned}
\frac{dA}{dt} &= \frac{1}{2}r^2\frac{d\theta}{dt} \\
\frac{d\theta}{dt} &= \frac{L}{mr^2} \\
\frac{dA}{dt} &= \frac{1}{2}r^2\frac{L}{mr^2} \\
\frac{dA}{dt} &= \frac{L}{2m}
\end{aligned}$$

where $\frac{L}{2m}$ is a constant, given conservation of momentum.

(c) The orbits of several stars have been measured orbiting the massive black hole near the center of our Galaxy. One of them has an orbital period of 15 years, and the orbital radius is 0.12 second of arc (as seen from the Earth). Take the distance to the Galactic center to be 8 kpc. Compute the mass of the black hole, starting from $F = ma$. Express your answer in units of the Sun's mass ($M_\odot$). (Assume that Newton's law of gravity is applicable for orbits sufficiently far from a black hole, and that the orbiting star satisfies this condition.

$$r \simeq 8000\,\text{pc} * \tan 0.12'' \simeq 1.4\times10^{14}\text{m}; \quad P \simeq 4.73\times10^8\text{s} \simeq 15\,\text{yrs}$$

$$\frac{GM_{\text{bh}}}{r^3} = \left(\frac{2\pi}{P}\right)^2$$

$$M_{\text{bh}} \simeq \frac{4\pi^2(1.44\times10^{14}\text{m})^3}{(7\times10^{-11}\text{N-m}^2/\text{kg}^2)(4.73\times10^8\text{s})^2} \simeq 3.7\times10^6 M_\odot$$

Problem 2 (25 points)

Geometric and Physical Optics

(a) Find the theoretical limiting angular resolution of a commercial 8-inch (diameter) optical telescope being used in visible (at $\lambda = 5000\mathring{A} = 500$ nm $= 5 \times 10^{-5}$ cm $= 5 \times 10^{-7}$ m).

$$\theta = 1.22\frac{\lambda}{D} = 1.22\frac{5 \times 10^{-5}\ \text{cm}}{8 \times 2.54\ \text{cm}} = 2.46 \times 10^{-6}\ \text{radians} = 0.49\ \text{arcsecs}$$

(b) A large ground-based telescope has an effective focal length of 10 meters. Two astronomical objects are separated by 1 arc second in the sky. How far apart will the two corresponding images be in the focal plane?

$$s = f\theta = 1000\ \text{cm} \times \frac{1}{2 \times 10^5}\ \text{radians} = 0.005\ \text{cm} = 50\,\mu\text{m}$$

(c) A collimated light beam propagating *in water* is incident on the surface (air/water interface) at an angle θ_{water} with respect to the surface normal. If the index of refraction of water is $n = 1.3$, find an expression for the angle of the light once it emerges from the water into the air, θ_{air}. What is the critical angle, i.e., $\theta_{\text{water}} = \theta_{\text{crit}}$, such that the light will not emerge from the water?

$$\begin{aligned} \text{By Snell's Law: } n_{water}\sin\theta_{water} &= n_{air}\sin\theta_{air} \\ \theta_{air} = \sin^{-1}(n_{water}\sin\theta_{water}) &= \sin^{-1}(1.3\sin\theta_{water}) \end{aligned}$$

$$\begin{aligned} \text{For critical incidence, the angle of refraction, } \theta_{air}, &= 90^\circ \\ \text{So, using Snell's Law, } 1.3\sin\theta_{water} &= 1 \times \sin 90^\circ = 1 \\ \text{Therefore, } \theta_{water} = \theta_{crit} = \sin^{-1}\left(\frac{1}{1.3}\right) &= 50.28^\circ \end{aligned}$$

Problem 3 (25 points)

Short Answer Questions

(a) Use the Bohr model of the atom to compute the wavelength of the transition from the $n = 100$ to $n = 99$ levels. [Useful relation: the wavelength of $L\alpha$ (n = 2 to n = 1 transition) is 1216Å.]

$$\begin{aligned}
E_n &= \frac{-13.605\text{eV}}{n^2} \\
\Delta E &= 13.605\text{eV}(1/99^2 - 1/100^2) = 2.76 \times 10^{-5}\text{eV} = h\nu \\
c &= \lambda\nu \\
\lambda &= \frac{hc}{\Delta E} \simeq 4.5 \times 10^8\text{Å} \simeq 4.5\,\text{cm}
\end{aligned}$$

-or-

$$\frac{1}{\lambda} = \frac{1}{\lambda_0}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

But we are given $\lambda = 1216$Å when $n_i = 2$ and $n_f = 1$. This implies that

$$\frac{1}{1216\,\text{Å}} = \frac{1}{\lambda_0}\left(\frac{1}{1^2} - \frac{1}{2^2}\right), \quad \text{or} \quad \lambda_0 = 912\,\text{Å}$$

Thus,

$$\frac{1}{\lambda} = \frac{1}{912\,\text{Å}}\left(\frac{1}{99^2} - \frac{1}{100^2}\right)$$

or, $\lambda \simeq 4.5 \times 10^8$ Å $\simeq 4.5$ cm.

(b) A galaxy moves directly away from us with speed v, and the wavelength of its Hα line is observed to be 6784Å. The rest wavelength of Hα is 6565Å. Find v.

$$\lambda \simeq \lambda_0(1 + v/c)$$

where $\lambda = 6784$Å and $\lambda_0 = 6565$Å. Rearranging,

$$\frac{v}{c} \simeq \frac{\lambda - \lambda_0}{\lambda_0} \simeq \frac{6784 - 6565}{6565} \Rightarrow v \simeq 0.033\,c$$

(c) A star is at a distance from the Earth of 300 pc. Find its parallax angle, π.

$$\begin{aligned}
D &= 1\text{pc}/\pi'' \\
\pi'' &= 1\text{pc}/300\text{pc} \\
\pi'' &= 0.003''
\end{aligned}$$

Problem 4 (20 points)

Luminosity and Magnitudes

(a) A particular star has an absolute magnitude M = -7. If this star is observed in a galaxy that is at a distance of 3 Mpc, what will its apparent magnitude be? What is the distance modulus to this galaxy?

$$\text{Given: } M = -7 \text{ and } d = 3 \text{ Mpc}$$

$$\text{Apparent Magnitude: } m = M + 5 \log\left[\frac{d}{10\text{pc}}\right] = -7 + 5\log\left[\frac{3\times 10^6}{10}\right] = 20.39$$

$$\text{Distance Modulus: } DM = m - M = 20.39 + 7 = 27.39$$

(b) The differential luminosity from a star, ΔL, with an approximate blackbody spectrum, is given by:

$$\Delta L = \frac{8\pi^2 c^2 R^2}{\lambda^5[e^{hc/(\lambda kT)} - 1]}\Delta\lambda$$

where R is the radius of the star, T is its effective surface temperature, and λ is the wavelength. ΔL is the power emitted by the star between wavelengths λ and $\lambda + \Delta\lambda$ (assume $\Delta\lambda \ll \lambda$).

(i) The star is at distance d. Find the star's spectral intensity $I(\lambda)$ at the Earth, where $I(\lambda)$ is defined as the power per unit area per unit wavelength interval.

(ii) Without working out all the arithmetic, show how you could determine the temperature T of the star by measuring the spectral intensity at two different wavelengths, e.g., $I(\lambda_1)$ and $I(\lambda_2)$. Neither the quantity d nor R should appear in your expression for T.

(i)

$$I(\lambda) = \frac{1}{4\pi d^2}\frac{\Delta L}{\Delta\lambda} = \frac{2\pi c^2 R^2}{\lambda^5[e^{hc/(\lambda kT)} - 1]d^2}$$

(ii) The two spectral intensities at two wavelengths, λ_1 and λ_2, are given by

$$I_1 = I(\lambda_1) = \frac{2\pi c^2 R^2}{\lambda_1^5[e^{hc/(\lambda_1 kT)} - 1]d^2}$$

$$I_2 = I(\lambda_2) = \frac{2\pi c^2 R^2}{\lambda_2^5[e^{hc/(\lambda_2 kT)} - 1]d^2}$$

So,

$$\frac{I_1}{I_2} = \frac{\lambda_2^5[e^{hc/(\lambda_2 kT)} - 1]}{\lambda_1^5[e^{hc/(\lambda_1 kT)} - 1]}$$

T is the only unknown in this equation and can therefore be determined.