8.282 X-Ray Reflection at Grazing Incidence

When X-rays ($0.1\,\mathring{A} \lesssim \lambda \lesssim 100\,\mathring{A}$) strike a typical metal surface at normal incidence, most of the radiation is absorbed. An X-ray entering the metal "sees" most of the electrons as free (i.e., not bound to the nuclei) since the energy of the X-ray (typically in the keV range) is much greater than the binding energy of all but the electrons within the innermost shells of the atom. The X-ray therefore finds itself effectively in a plasma with electron number density

$$N_e \simeq \frac{(Z-2)\rho}{A\, m_p} \quad \text{electrons cm}^{-3} \qquad (1)$$

where Z and A are the atomic number and weight of the metal, ρ is the mass density, and m_p is the mass of the proton. The presence of the plasma leads to an index of refraction which can be calculated from first principles (at the 8.03 level):

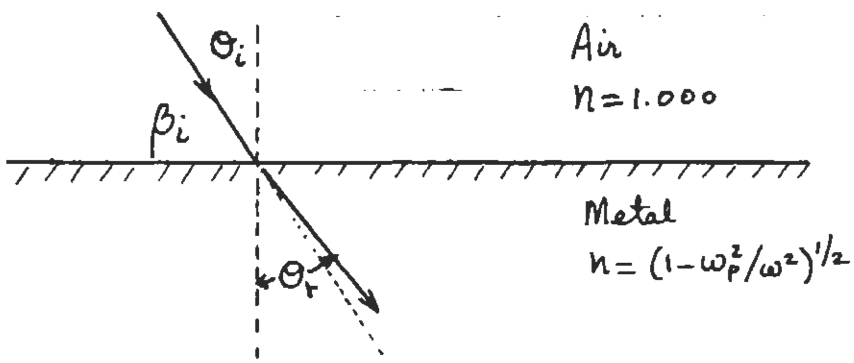
$$n \simeq \left(1 - \omega_p^2/\omega^2\right)^{1/2} \qquad (2)$$

where ω_p is the so-called "plasma frequency" given by the following constants: $\omega_p^2 = \dfrac{4\pi N_e e^2}{m_e}$, and ω is the frequency of the X-radiation.

[e is the charge on the electron and m_e is the mass of an electron.]

The few remaining strongly bound electrons contribute to the absorption of X-rays via the photoelectric effect. This process results in an imaginary part to the index of refraction that we have neglected in equation (2).

Consider now a beam of X-rays incident on a metal plate as shown in the sketch.



The X-rays that penetrate the metal will have an angle of refraction θ_r given by Snell's law:

$$\sin\theta_i = n\sin\theta_r \cong (1-\omega_p^2/\omega^2)^{1/2}\sin\theta_r \qquad (3)$$

Note that since the index in the metal is <u>less than</u> unity the rays bend <u>away</u> from the normal → in contrast to the more familiar case of light entering a piece of glass. However, the X-rays that enter the metal are quickly absorbed and the angle of refraction is of little consequence.

An interesting phenomenon occurs when θ_i approaches 90°. At some point, θ_r will reach 90° and the incident radiation will undergo total "internal" reflection. (Note, however, that this should more properly be called total "external" reflection.). The critical angle of incidence for total internal reflection is given by:

$$\sin\theta_{crit} \simeq (1-\omega_p^2/\omega^2)^{1/2} \tag{4}$$

At this point, it is more convenient to use the complementary "grazing angle" β_i instead of the angle of incidence θ_i. Rewriting eq. (4), we find

$$\cos\beta_{crit} \simeq (1-\omega_p^2/\omega^2)^{1/2} \tag{5}$$

As we shall show below, $\omega >> \omega_p$, and we can therefore expand both sides of eq. (5) in a Taylor expansion to yield

$$1-\frac{\beta_{crit}^2}{2}+\cdots \simeq 1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}+\cdots \tag{6}$$

This leads to our expression for the critical grazing angle

$$\boxed{\beta_{crit} \simeq \frac{\omega_p}{\omega}} \tag{7}$$

We can now evaluate the constant ω_p from eq. (1) and the expression for ω_p given two lines below eq. (2).

$$\omega_p^2 \cong \frac{4\pi(Z-2)\rho N e^2}{A m_e} \simeq 1.9\times10^{33}\,\frac{(Z-2)\rho}{A} \tag{8}$$

For a typical metal such as Fe

$$\omega_p \simeq 8\times10^{16} \text{ radians/sec} \tag{9}$$

The frequency of the X-rays is given by

$$\omega = \frac{2\pi c}{\lambda} \simeq 1.9 \times 10^{19} \left(\frac{\lambda}{\text{Å}}\right)^{-1} \text{ radians/sec} \tag{10}$$

where λ is expressed in Ångströms.

Combining expressions (9) and (10) we find

$$\boxed{\beta_{crit} \simeq \frac{\omega_p}{\omega} \simeq 0.0043 \left(\frac{\lambda}{\text{Å}}\right)} \tag{11}$$

Energy (keV)	λ (Å)	β_{crit} (degrees)
12	1	0.3
2.5	5	1.5
1.2	10	3
0.6	20	6

Thus, to build an imaging X-ray telescope that operates over the energy range of a fraction of a keV to a few keV, the incident X-rays must be reflected at grazing angles of about a degree or so.