8.321 Quantum Theory-I Fall 2017

Prob Set 10

1. (a) Verify that the spin-1/2 operators

$$\vec{S} = \frac{\hbar\vec{\sigma}}{2} \tag{1}$$

satisfy the spin commutation relations

$$[S_i, S_j] = \hbar\epsilon_{ijk}S_k \tag{2}$$

(b) Construct the operators for finite rotations about the three coordinate axes (x, y, x) of a spin-1/2 system and write these as 2×2 matrices in the basis of eigenstates of S_z.

(c) Consider the eigenstate $|+x\rangle$ of S_x with eigenvalue $+\frac{\hbar}{2}$. Construct a new state

$$|\psi\rangle = \mathcal{D}_z(\phi)|+x\rangle \tag{3}$$

obtained by operating on this with a rotation about the z-axis by angle ϕ. Show that this is an eigenstate of a spin operator $\vec{S}\cdot\hat{n}$ and find the unit vector $\hat{n}$.

2. Given an $SU(2)$ matrix U show that the quantitates

$$R_{ij} = \frac{1}{2}Tr\left(\sigma_j U^\dagger \sigma_i U\right) \tag{4}$$

form the elements of an $SO(3)$ rotation matrix.

This explicitly shows that $(U, -U)$ correspond to the same $SO(3)$ matrix.

3. (a) Use the result of Prob 4 in Problem Set 9 to study a finite rotation about, say the x axis, by an angle θ in a spin-1 system. Show that

$$\mathcal{D}(R(\theta,\hat{x}) = 1 - i\sin\theta\frac{S_x}{\hbar} + (\cos\theta - 1)\left(\frac{S_x}{\hbar}\right)^2 \tag{5}$$

   (b) A spinless particle in a spherically symmetric potential is in a state with orbital angular momentum $l = 1, m = 1$. The state is rotated by an angle θ about the x axis. What is the probability that a measurement of L_z will yield the value $m = 1$? Repeat with a rotation by θ in the z (instead of x) direction.

4. (a) Consider an electron in an atom in a state of orbital angular momentum l. Determine the allowed values of the total angular momentum J of the electron obtained by adding the orbital and spin angular momenta.

   (b) Find the Clebsch-Gordon coefficients describing the change from the basis $|lm_l, m_s\rangle$ to the basis $|J, m, l\rangle$. Here m_l, m_s, m are the eigenvalues of the orbital, spin, and total angular momenta along the z axis.

5. **Sakurai 4.2**

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