8.321 Quantum Theory-I Fall 2017

Midterm Quiz

Oct 18, 2017

Useful facts.

$$\int du e^{-au^2} = \sqrt{\frac{\pi}{a}} \tag{1}$$

Ground state wave function of a 1d simple harmonic oscillator of mass m, frequency ω is

$$\begin{aligned} \psi_0(x) &= N_0 e^{-\frac{x^2}{2x_0^2}} \\ x_0 &= \sqrt{\frac{\hbar}{m\omega}} \\ N_0 &= \frac{1}{\pi^{\frac{1}{4}}\sqrt{x_0}} \end{aligned}$$

1. **(25 points)**

   (a) We showed in class that Hermitian operators can always be diagonalized. Prove here that an anti-hermitian operator can always be diagonalized. What properties do its eigenvalues satisfy?

   (b) For any unitary operator U show that the operators $U + U^\dagger$ and $U - U^\dagger$ commute with each other. Show that this implies that unitary operators can always be diagonalized.

   (c) What properties do the eigenvalues of a unitary operator satisfy?

   (d) If O is a Hermitian operator show that e^{iO} is unitary.

2. **25 points**

Consider a particle moving in one dimension with a potential $V(x)$; the Hamiltonian is

$$H = \frac{p^2}{2m} + V(x) \tag{2}$$

Let $|n\rangle$ be the (normalized) energy eigenstates of H.

Now suppose the potential is changed suddenly at time $t = 0$ so that the new Hamiltonian is

$$H' = \frac{p^2}{2m} + V(x-a) \tag{3}$$

Let $|n'\rangle$ be the energy eigenstates of H'.

(a) Define an operator O_a such that $O_a|n\rangle = |n'\rangle$ for all n. On general grounds what can you say about O_a? Find an explicit expression for O_a in terms of the momentum operator p.

(b) Suppose that for $t < 0$ the particle was in the ground state $|0\rangle$ of H. Assume that at $t = 0^+$ it stays in the same state. In terms of the operator O_a and the states $|n\rangle$, what is the probability that at $t = 0^+$ it is in the ground state $|0'\rangle$ of H'?

(c) Re-express your answer for the probability in the previous part as an integral over momentum and the momentum space wavefunction $\langle p|0\rangle$. Evaluate explicitly for the simple harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$.

3. **(25 points)**

A quantum particle with charge e and mass m moves in a time dependent electric field $E = E(t)$ in one dimension. The function $E(t)$ takes the values

$$\begin{aligned} E(t) = &\quad 0 \qquad -\infty < t < 0 &(4)\\ = &\quad E_0 \qquad 0 \le t < t_0 &(5)\\ = &\quad 0 \qquad t_0 < t < \infty &(6)\end{aligned}$$

Assume that at $t = -\infty$ the particle is in a momentum eigenstate with $p(t = -\infty) = p_0$.

(a) What is the expectation value $\langle p \rangle$ of the momentum at time $t = +\infty$? What is $\langle (\Delta p)^2 \rangle$ at $t = \infty$?

(b) Find the change in the expectation value of the energy between $t = \infty$ and $t = -\infty$.

4. **Ramsey interference (25 points)**

A powerful protocol for manipulating a two level system is a set-up known as a Ramsey interferometer. Consider an atom with two internal states which we label $|\downarrow\rangle$ (the ground state g) and $|\uparrow\rangle$ (the excited state e). These two states have an energy splitting that we denote $\hbar\omega_0$. The atom is initially in the ground state. The atom is then subject to an oscillating external field that causes transitions between the two internal levels. A suitable Hamiltonian to describe this system is

$$H_0 = \frac{\hbar\omega_0}{2}\sigma^z \tag{7}$$

The atom is initially in the ground state. The atom is then subject to a time dependent external field that adds a term $V(t)$ to the Hamiltonian of the form

$$\begin{aligned} V(t) &= \tfrac{\hbar\Omega}{2}\left(\cos(\omega t)\sigma^x + \sin(\omega t)\sigma^y\right) \qquad 0 < t < \tau & (8)\\ &= 0 \qquad \tau < t < T + \tau & (9)\\ &= \tfrac{\hbar\Omega}{2}\left(\cos(\omega t)\sigma^x + \sin(\omega t)\sigma^y\right) \qquad T + \tau < t < T + 2\tau & (10)\end{aligned}$$

(a) The time τ is chosen to correspond to what is known as a $\frac{\pi}{2}$ pulse. At the end of the first $\frac{\pi}{2}$ pulse suppose the atom is in the state $\frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle)$. If the atom were instead prepared in the excited state e at $t = 0$ what state will it be in after the first $\frac{\pi}{2}$ pulse?

(b) Returning to the original situation where the atom is initially in the ground state, what state is it in at time $T + \tau$?

(c) Find the probability that at the end ($i.e$ at time $t = T + 2\tau$), the atom is in the ground state g.