Lecture 14 (Oct. 30, 2017)

14.1 Magnetic Monopoles

Last time, we considered a magnetic field with a magnetic monopole configuration, and began to approach describing the quantum mechanics of a charged particle moving in such a magnetic field. The presence of the monopole violates the condition $\nabla \boldsymbol{B} = 0$, which means that we cannot globally define a vector potential $\boldsymbol{A}$ such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

We began by considering an isolated magnetic monopole of strength g located at the origin, which gives a magnetic field

$$\boldsymbol{B} = \frac{g}{r^2}\hat{\boldsymbol{e}}_r \,. \tag{14.1}$$

Consider a sphere of radius r, and a closed circular contour C at θ on the sphere. The magnetic flux through the upper cap Σ_+ bounded by C is

$$\Phi_C = 2\pi g(1 - \cos\theta)\,. \tag{14.2}$$

If we choose

$$\boldsymbol{A} = A_\phi \hat{\boldsymbol{e}}_\phi \,, \tag{14.3}$$

then we have

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\ell} = 2\pi r \sin\theta A_\phi = \int_{\Sigma_+} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 2\pi g(1 - \cos\theta)\,, \tag{14.4}$$

which gives us

$$A_\phi = \frac{g(1 - \cos\theta)}{r \sin\theta}\,. \tag{14.5}$$

Where is this potential well-defined? We see that it diverges at $r = 0$, but we expected this, because the magnetic field is divergent at $r = 0$. As $\theta \to 0$, the cosine in the numerator approaches 1 more quickly than the sine in the denominator approaches 0, so A_ϕ is well-defined at $\theta = 0$. However, it blows up at $\theta = \pi$. Let us rename this vector potential, as

$$A_\phi^+ = \frac{g(1 - \cos\theta)}{r \sin\theta}\,. \tag{14.6}$$

In calculating this vector potential, we made the arbitrary choice to consider the magnetic flux through the upper cap bounded by C instead of the lower cap. If we instead consider the flux through the lower cap, we would find a vector potential $\boldsymbol{A} = A_\phi^- \hat{\boldsymbol{e}}_\phi$, where

$$A_\phi^- = -\frac{g(1 + \cos\theta)}{r \sin\theta}\,. \tag{14.7}$$

This vector potential is not well-defined at $\theta = 0$, but is well-defined at $\theta = \pi$.

Thus, we cannot use A_ϕ^+ at the south pole, and we cannot use A_ϕ^- at the north pole, but everywhere else we could equally well choose either expression for the vector potential. Because both of these vector potentials describe the same electromagnetic fields in these regions, they must differ by the gradient of some function. Indeed, we see that

$$\boldsymbol{A}^+ - \boldsymbol{A}^- = \left(A_\phi^+ - A_\phi^-\right)\hat{\boldsymbol{e}}_\phi = \frac{2g}{r \sin\theta}\hat{\boldsymbol{e}}_\phi = 2g\nabla\phi\,. \tag{14.8}$$

Thus, as expected, these two vector potentials differ from one another by a gauge transformation.

We have found that, in order to describe the magnetic monopole field in terms of vector potentials, we can write $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where

$$\boldsymbol{A} = \begin{cases} \boldsymbol{A}^+ = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\boldsymbol{e}}_\phi \,, & \text{for } 0 \leq \theta < \frac{\pi}{2} + \epsilon \,, \\ \boldsymbol{A}^- = -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{\boldsymbol{e}}_\phi \,, & \text{for } \frac{\pi}{2} - \epsilon < \theta \leq \pi \,, \end{cases} . \tag{14.9}$$

In the region where $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$ equal $\boldsymbol{A}$, they are well-defined. In the overlap region, the differ by a gauge transformation.

14.1.1 QM of a Charged Particle Moving in a Magnetic Monopole Field

Consider a particle of electric charge e. For $0 \leq \theta < \frac{\pi}{2} + \epsilon$, let the wavefunction be $\psi^+(r, \theta, \phi)$, and for $\frac{\pi}{2} - \epsilon < \theta \leq \pi$, let the wavefunction be $\psi^-(r, \theta, \phi)$. Assume that we have already determined these by solving the Schrödinger equation. In the overlap region, ψ^+ and ψ^- must be related by a gauge transformation. We saw in a previous lecture that if we make a gauge transformation

$$\boldsymbol{A}^+ = \boldsymbol{A}^- + 2g\nabla\phi \,, \tag{14.10}$$

then the wavefunction must change by

$$\psi^+ = \psi^- e^{2ieg\phi/\hbar c} \,. \tag{14.11}$$

Now, we note that the wavefunctions ψ^+ and ψ^- must be single-valued when $\phi \to \phi + 2\pi$. This requires that

$$\frac{2eg}{\hbar c} = n \in \mathbb{Z} \,, \tag{14.12}$$

i.e.,

$$g = \frac{nhc}{2\pi e} \,, \quad n \in \mathbb{Z} \,. \tag{14.13}$$

This is the *Dirac quantization condition*: the magnetic monopole field strength must be quantized, with the quantum a function of the electric charge.

We can gain intuition about this result with a vague classical analogue. Note, however, that this quantization is a purely quantum effect, and so cannot be fully explained classically. Let's consider the charged particle moving in the field of an magnetic monopole from a classical point of view. Consider an electric monopole of strength e and a magnetic monopole of strength g, both static, and displaced from one another by distance d along the z-axis. We can then ask about the total angular momentum stored in the electromagnetic field; this information is contained in the Poynting vector, which is proportional to $\boldsymbol{E} \times \boldsymbol{B}$. Purely by symmetry, we conclude that the angular momentum must be directed along the z-axis. If we carry out the calculation, we find that the total angular momentum is independent of the distance d, and is proportional to eg. In quantum mechanics, we know that angular momentum is quantized. If we require the total angular momentum we found to be an integer multiple of $\frac{\hbar}{2}$, then we recover the Dirac quantization condition. Incidentally, this gives us a model of what is "spinning" in a spin-$\frac{1}{2}$ particle: a bound state of a bosonic magnetic monopole and a bosonic electric monopole has spin-$\frac{1}{2}$. The naïve statistics of this bound state would be bosonic, because both the electric and magnetic monopoles are bosonic; however, the interactions between the two charges lead to a change in the statistics.

14.2 Charged Particle in a Uniform Magnetic Field

Consider a particle of electric charge e moving in a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ in three dimensions. The Hamiltonian is

$$H = \frac{\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2}{2m}\,. \tag{14.14}$$

Only $B_z \neq 0$, and so we can always choose $A_z = 0$ and A_x, A_y independent of z. The Hamiltonian then becomes

$$H = \frac{p_z^2}{2m} + \frac{\Pi_x^2 + \Pi_y^2}{2m}\,, \tag{14.15}$$

where

$$\Pi_{x,y} = p_{x,y} - \frac{e}{c}A_{x,y} \tag{14.16}$$

are the kinematic momenta in the x- and y-directions. Note that $[p_z, H] = 0$, so we can label the eigenstates by p_z. We can then write the Hamiltonian in the form

$$H = \frac{p_z^2}{2m} + H_{2\text{d}}\,, \tag{14.17}$$

with

$$H_{2\text{d}} = \frac{\Pi_x^2 + \Pi_y^2}{2m}\,, \tag{14.18}$$

and we only have to determine the spectrum of $H_{2\text{d}}$.

The trick is to notice that Π_x and Π_y have a simple commutation relation,

$$[\Pi_x, \Pi_y] = \left[-i\hbar\partial_x - \frac{e}{c}A_x, -i\hbar\partial_y - \frac{e}{c}A_y\right] = \frac{eB}{c}i\hbar\,. \tag{14.19}$$

Thus, these two kinematic momenta (appropriately rescaled) are canonically conjugate variables, and the Hamiltonian $H_{2\text{d}}$ looks like the sum of squares of canonically conjugate variables, which is the Hamiltonian of the simple harmonic oscillator in one dimension. More precisely, let

$$X = \frac{c\Pi_x}{eB}\,, \quad P = \Pi_y\,, \tag{14.20}$$

Then, $[X, P] = i\hbar$, and

$$H_{2\text{d}} = \frac{P^2}{2m} + \frac{1}{2m}\left(\frac{eB}{c}\right)^2 X^2 = \frac{P^2}{2m} + \frac{1}{2}m\omega_c^2 X^2\,, \tag{14.21}$$

with

$$\omega_c = \frac{eB}{mc}\,. \tag{14.22}$$

This is the one-dimensional SHO Hamiltonian, with frequency ω_c, known as the *cyclotron frequency*.

As an aside, the classical motion of a charged particle a uniform magnetic field is described by circular orbits in a plane orthogonal to the magnetic field. Matching the centrifugal force with the force from the magnetic field, we have

$$\frac{mv^2}{R} = \frac{evB}{c}\,, \tag{14.23}$$

which gives a radius of

$$R = \frac{mvc}{eB}\,, \tag{14.24}$$

known as the *cyclotron radius*. The time period of the orbit is

$$T = \frac{2\pi R}{v} = \frac{2\pi}{\omega_c}, \tag{14.25}$$

with ω_c the cyclotron frequency. This is the classical origin of the cyclotron frequency.

Now, we have the Hamiltonian

$$H_{2\mathrm{d}} = \frac{P^2}{2m} + \frac{1}{2}m\omega_c^2 X^2 \tag{14.26}$$

with $[X, P] = i\hbar$. We can immediately conclude that the energy levels are

$$E_n^{(2\mathrm{d})} = \hbar\omega\left(n + \frac{1}{2}\right). \tag{14.27}$$

The three-dimensional energy levels of the full Hamiltonian H are then

$$E_n^{(3\mathrm{d})}(p_z) = \frac{p_z^2}{2m} + \hbar\omega\left(n + \frac{1}{2}\right). \tag{14.28}$$

However, we are not done, because we do not know the degeneracies of these energy levels. We will find that the spectrum is highly degenerate.

14.2.1 Degeneracy

Why is there degeneracy in the spectrum? One way to understand the degeneracy is to notice that we can define new coordinates in the problem,

$$R_x := x + \frac{c}{eB}\Pi_y, \quad R_y := y - \frac{c}{eB}\Pi_x. \tag{14.29}$$

Note that $[R_x, R_y] = -\frac{c}{eB}i\hbar$. Thus, R_x and R_y are canonically conjugate up to a multiplicative factor. Furthermore, we note that

$$\begin{aligned}[R_x, \Pi_x] &= [x, \Pi_x] + \frac{c}{eB}[\Pi_y, \Pi_x] \\ &= i\hbar - \frac{c}{eB}\left(\frac{eB}{c}i\hbar\right) \\ &= 0.\end{aligned} \tag{14.30}$$

Similarly, we find that $[R_x, \Pi_y] = 0$, and more generally,

$$[R_i, \Pi_j] = 0. \tag{14.31}$$

Thus,

$$[R_i, H] = 0. \tag{14.32}$$

We have two operators that each commute with the Hamiltonian, but they do not commute with one another. Recall from a previous homework that when this is the case, the Hamiltonian must be degenerate.

What is the physical meaning of these coordinates R_i? Recall that classically, the particle undergoes circular motion in the presence of the uniform magnetic field. Classically, the vector $\boldsymbol{R} = (R_x, R_y)$ is the center of the cyclotron orbit: if (x, y) are the time-dependent coordinates of the particle moving in its circular orbit, and we take $\Pi_i = mv_i$, then (R_x, R_y) are the time-independent coordinates of the center of the orbit. This point is called the *guiding center*. We see that in quantum mechanics, the coordinates of the guiding center do not commute with one another in the presence of the magnetic field. The size of the cyclotron orbit will be fixed such that the magnetic flux through the orbit yields one of the quantized energies, but the location of the orbit is not fixed, which leads to the degeneracy. These degenerate energy levels are called *Landau levels*.