Lecture 15 (Nov. 1, 2017)

15.1 Charged Particle in a Uniform Magnetic Field

Last time, we discussed the quantum mechanics of a charged particle moving in a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. We decomposed the Hamiltonian as

$$H = \frac{p_z^2}{2m} + H_{\text{2d}}\,, \tag{15.1}$$

with

$$H_{\text{2d}} = \frac{\Pi_x^2 + \Pi_y^2}{2m\,.} \tag{15.2}$$

The Hamiltonian H_{2d} has spectrum

$$E_n^{(\text{2d})} = \hbar\omega_c\left(n + \frac{1}{2}\right), \tag{15.3}$$

with the cyclotron frequency given by

$$\omega_c = \frac{eB}{mc}\,. \tag{15.4}$$

We then wanted to determine the degeneracy of these energy levels, called *Landau levels.* We defined the *guiding center coordinates*,

$$R_x := x + \frac{c}{eB}\Pi_y\,, \quad R_y := y - \frac{c}{eB}\Pi_x\,. \tag{15.5}$$

These variables are conjugate, up to a multiplicative factor,

$$[R_x, R_y] = -\frac{c}{eB}i\hbar = -i\ell_B^2\,, \tag{15.6}$$

and both commute with the Hamiltonian. As we showed on the homework, this implies that the Hamiltonian is degenerate. Here, ℓ_B is the *magnetic length*, which is the length scale such that if we cut out a circular disk of radius ℓ_B orthogonal to the magnetic field, then the flux through the disk is on the order of one flux quantum:

$$\pi B\ell_B^2 = \frac{hc}{2e} \sim \frac{hc}{e} = \Phi_0\,, \tag{15.7}$$

where Φ_0 is the flux quantum.

The guiding center coordinates are the coordinates of the center of the cyclotron orbit, which are a constant of the motion. This statement is equivalent to the statement that the guiding center coordinates commute with the Hamiltonian. In the quantum theory, the size of the orbit is set by the only length scale in the problem, the magnetic length. We expect that the degeneracy is the number of flux quanta passing through the sample, $\sim \frac{BA}{\Phi}$, where A is the area of the sample.

Now we will calculate the degeneracy explicitly. We pick a particular gauge to work in, for convenience. We choose

$$A_y = Bx\,, \quad A_x = 0\,. \tag{15.8}$$

This is known as *Landau gauge.* In this gauge,

$$\begin{aligned} H_{\text{2d}} &= \frac{p_x^2}{2m} + \frac{\left(p_y - \frac{eBx}{c}\right)^2}{2m} \\ &= \frac{p_x^2}{2m} + \frac{1}{2}m\omega_c^2\left(x - \frac{cp_y}{eB}\right)^2. \end{aligned} \tag{15.9}$$

In this gauge, $[p_y, H] = 0$, so we can label states by their p_y eigenvalue. For any fixed p_y, we get a one-dimensional simple harmonic oscillator of frequency ω_c, which reproduces the spectrum

$$E_n^{(2\mathrm{d})} = \hbar\omega_c\left(n + \frac{1}{2}\right). \tag{15.10}$$

The energy eigenfunctions take the form

$$\psi_{n,p_y} = e^{ip_y y/\hbar}\phi_n\left(x - \frac{cp_y}{eB}\right). \tag{15.11}$$

This is a plane wave multiplied by a shifted SHO wavefunction. It is now clear that states with different values of p_y will be degenerate. The degeneracy question then becomes the question of how many distinct values of p_y exist for a system of a given area.

Consider a sample of lengths L_x and L_y along the x- and y-directions, respectively, with periodic boundary conditions along the y-direction. We then must have

$$e^{ip_y L_y/\hbar} = 1\,, \tag{15.12}$$

which gives us quantized momentum modes along the y-direction,

$$p_y = \frac{2\pi m\hbar}{L_y}\,, m \in \mathbb{Z}\,. \tag{15.13}$$

Each of the degenerate wavefunctions $\psi_{n,p_y}(x, y)$ has a modulus $\left|\phi_n\left(x - \frac{cp_y}{eB}\right)\right|$ and is centered at

$$x = \left(\frac{c}{eB}\right)\left(\frac{2\pi m\hbar}{L_y}\right). \tag{15.14}$$

However, we cannot arbitrarily shift these SHO groundstate wavefunctions, because the sample has a finite length L_x in the x-direction. Thus, the y-direction momentum is quantized because of the finite length in the y-direction, and has a finite number of possible values because of the finite length in the x-direction. The number of independent states is then

$$g = \frac{L_x}{\left(\frac{c}{eB}\right)\left(\frac{2\pi\hbar}{L_y}\right)} = \frac{eBL_xL_y}{2\pi\hbar c} = \frac{eBA}{hc} = \frac{BA}{\Phi_0}\,, \tag{15.15}$$

as we expected.

In classical mechanics, we can prove that there is no diamagnetism. In classical statistical mechanics, everything is determined by the partition function,

$$Z \propto \int \prod \frac{\mathrm{d}x_i\,\mathrm{d}p_i}{(2\pi\hbar)^3}\, e^{-\beta\left(\sum_i \frac{\boldsymbol{p}_i^2}{2m} + V(\{\boldsymbol{x}_i\})\right)}. \tag{15.16}$$

In the presence of a magnetic field, the Hamiltonian changes,

$$Z[\boldsymbol{A}] \propto \int \prod \frac{\mathrm{d}x_i\,\mathrm{d}p_i}{(2\pi\hbar)^3}\, e^{-\beta\left(\sum_i \frac{(\boldsymbol{p}_i - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x}_i))^2}{2m} + V(\{\boldsymbol{x}_i\})\right)}. \tag{15.17}$$

However, we can simply shift the momenta as $\boldsymbol{p}_i' = \boldsymbol{p}_i - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x}_i)$, which then gives us $Z[A] = Z[A = 0]$. Thus, classically, the magnetic field does not affect the statistical mechanics. However, we have just seen that in quantum mechanics, there is an effect. This effect is purely quantum mechanical.

We might wonder how a two-dimensional problem of motion in the x, y-plane turned into a one-dimensional SHO. The electron is localized around the guiding center coordinates. However, we see that the two coordinates do not commute with one another. Thus, there is only one real coordinate in the problem, and the other coordinate is its conjugate variable. This is why we are only solving a one-dimensional problem at the end of the day. Note that because R_x and R_y do not commute, we cannot be in a simultaneous eigenstate of both, so the location of guiding center itself must be smeared out.

15.2 Composite Systems

We now consider composite systems, which are systems composed of two (or more) subsystems. Suppose we have two systems A and B, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. We want to obtain the Hilbert space of the composite space $A+B$, which is called the *tensor product* of $\mathcal{H}_A$ and $\mathcal{H}_B$.

Consider a basis $|\{\phi_i\}\rangle$ of $\mathcal{H}_A$ and a basis $|\{\chi_j\}\rangle$ of $\mathcal{H}_B$. We will define new vectors $|\phi_i\rangle \otimes |\chi_j\rangle$ that live in a new Hilbert space, denoted $\mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B$. These $|\phi_i\rangle \otimes |\chi_j\rangle$ will provide a basis for the Hilbert space $\mathcal{H}_{A+B}$, meaning that any state in $\mathcal{H}_{A+B}$ can be decomposed as

$$|\psi_{A+B}\rangle = \sum_{i,j} \psi_{ij} |\phi_i\rangle \otimes |\chi_j\rangle \,. \tag{15.18}$$

Often times, we will omit the tensor product symbol $\otimes$ for convenience, when the meaning is clear. Thus, we will write $|\phi_i\rangle|\chi_j\rangle = |\phi_i\rangle \otimes |\chi_j\rangle$.

We can then write down operators on $\mathcal{H}_{A+B}$ and formulate the quantum theory as we have before. There are classes of operators that act on only one of the two subsystems, such as $\mathcal{O} = \mathcal{O}_A \otimes \mathbb{1}_B$, but in general we can have operators that act nontrivially on both subsystems at once.

Consider, in particular, a system of two spin-$\frac{1}{2}$ particles A and B. Some possible states in the composite Hilbert space can be written as a single tensor product, such as

$$|{\uparrow_A}\rangle \otimes |{\uparrow_B}\rangle \,, \quad |{\downarrow_A}\rangle \otimes |{\uparrow_B}\rangle \,, \quad |{\uparrow_A}\rangle \otimes \left[\frac{1}{\sqrt{2}}(|{\uparrow_B}\rangle + |{\downarrow_B}\rangle)\right]. \tag{15.19}$$

There are clearly an infinite number of such states. There are also more interesting states (entangled states) that cannot be written as a single tensor product, such as

$$\frac{1}{\sqrt{2}}(|{\uparrow_A}\rangle \otimes |{\downarrow_B}\rangle - |{\downarrow_A}\rangle \otimes |{\uparrow_B}\rangle)\,. \tag{15.20}$$

Some of the operators on this Hilbert space act only on one of the subsystems, such as $\sigma_A^z \otimes \mathbb{1}_B := \sigma_A^z$ (the second way of writing this is a notational convenience). However, there are more general operators that act nontrivially on both subsystems simultaneously, such as $\sigma_A^x \otimes \sigma_B^y := \sigma_A^x \sigma_B^y$.

15.2.1 Quantum Entanglement

Two subsystems A and B of a quantum mechanical system can be entangled with one another. For example, consider again the system of two spin-$\frac{1}{2}$ particles A and B. Some states, such as

$$|{\uparrow_A}\rangle \otimes |{\downarrow_B}\rangle \,, \quad |{\uparrow_A}\rangle \otimes |{\downarrow_B}\rangle \,, \tag{15.21}$$

have the property that if we restrict our view to only one of the subsystems, then the system is in a well-defined state of that subsystem. By contrast, if we consider a state such as

$$\frac{1}{\sqrt{2}}(|{\uparrow_A}\rangle \otimes |{\downarrow_B}\rangle \pm |{\downarrow_A}\rangle \otimes |{\uparrow_B}\rangle)\,, \tag{15.22}$$

then we see that neither spin by itself is in a definite quantum state, despite the fact that the composite system is in a definite state. This is an *entangled state*.

Entanglement is a statement about the relation of subsystems of a system to one another. It does not make sense to ask if the state of a system "is entangled." Instead, we can make statements about whether two subsystems are entangled with one another. If we have subsystems A and B of some composite system, then an unentangled state is precisely one that can be factored as a tensor product of states of the two subsystems,

$$|\psi_{A+B}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \,. \tag{15.23}$$

An entangled state is one that cannot be factorized in this way.