Lecture 7 (Sep. 27, 2017)

7.1 Spin Precession In a Magnetic Field

Last time, we began discussing the classic example of precession of a spin-$\frac{1}{2}$ particle in a magnetic field. The Hamiltonian of this system is

$$H = -\frac{ge}{2m}\boldsymbol{S}\cdot\boldsymbol{B}\,. \tag{7.1}$$

With $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, this becomes

$$H = -\frac{ge}{2m}S^z B\,. \tag{7.2}$$

The energy levels are

$$E_\pm = \mp\frac{ge\hbar B}{4m}\,, \tag{7.3}$$

giving a level splitting of

$$\hbar\omega = \left|\frac{ge\hbar B}{2m}\right|. \tag{7.4}$$

We define

$$\omega := \frac{g|eB|}{2m}\,, \tag{7.5}$$

so that the Hamiltonian can be written simply as

$$H = \omega S^z\,. \tag{7.6}$$

7.1.1 Schrödinger Picture

The time-evolution operator can be expressed in the energy eigenbasis (which coincides with the S^z eigenbasis) as

$$U(t,0) = e^{-iHt/\hbar} = e^{-i\omega S^z t/\hbar}\,. \tag{7.7}$$

Suppose that we have an arbitrary initial state

$$|\psi\rangle = c_+|+\rangle + c_-|-\rangle\,. \tag{7.8}$$

Using the time-evolution operator, we find that

$$\begin{aligned}|\psi(t)\rangle &= U(t,0)|\psi\rangle \\ &= e^{-i\omega S^z t/\hbar}|\psi\rangle \\ &= e^{-i\omega t/2}c_+|+\rangle + e^{i\omega t/2}c_-|-\rangle\,.\end{aligned} \tag{7.9}$$

We now know the state of the system at all times. For example, if we initially have $|\psi\rangle = |+\rangle$, then

$$|\psi(t)\rangle = e^{-i\omega t/2}|+\rangle\,, \tag{7.10}$$

which has

$$\mathrm{Prob}\big(S^z = \tfrac{\hbar}{2}\big) = 1 \tag{7.11}$$

for all times. This is why energy eigenstates are often called *stationary states*. Because time evolution is generated by the Hamiltonian, energy eigenstates do not change under time evolution (up to an unphysical overall phase).

Consider instead the case where the initial state is the spin-up eigenstate of S^x, i.e.,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\,. \tag{7.12}$$

Then we have

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\Big(e^{-i\omega t/2}|+\rangle + e^{i\omega t/2}|-\rangle\Big)\,. \tag{7.13}$$

Thus, after a time t we have

$$\begin{aligned}\text{Prob}\big(S^x = \tfrac{\hbar}{2} \text{ at time } t\big) &= |\langle S^x_+|\psi(t)\rangle|^2 \\ &= \frac{1}{2}|(\langle +| + \langle -|)|\psi(t)\rangle|^2 \\ &= \cos^2\left(\frac{\omega t}{2}\right)\,.\end{aligned} \tag{7.14}$$

Similarly,

$$\text{Prob}\big(S^x = -\tfrac{\hbar}{2} \text{ at time } t\big) = \sin^2\left(\frac{\omega t}{2}\right)\,. \tag{7.15}$$

We can check that this is true, but we know it must be true by conservation of probability. We then have

$$\langle S^x\rangle = \frac{\hbar}{2}\left[\cos^2\left(\frac{\omega t}{2}\right) - \sin^2\left(\frac{\omega t}{2}\right)\right] = \frac{\hbar}{2}\cos(\omega t)\,. \tag{7.16}$$

We see that this oscillates as a function of time, with angular frequency ω, as we expect from our classical intuition. What can we learn from this calculation? If we have a general initial state represented by $\hat{\boldsymbol{n}}$ on the Bloch sphere, then $\hat{\boldsymbol{n}}$ will precess around $\boldsymbol{B}$ with angular frequency ω. You will show this on the homework.

7.1.2 Heisenberg Picture

We can carry out this same calculation in the Heisenberg picture. In this picture, the states do not change as a function of time, but rather the operators do. The spin operator in the Heisenberg picture evolves according to the equation

$$\boldsymbol{S}(t) = e^{i\omega S^z t/\hbar}\boldsymbol{S}e^{-i\omega S^z t/\hbar}\,. \tag{7.17}$$

Taking the time derivative of this expression, we find

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \frac{i\omega}{\hbar}e^{i\omega S^z t/\hbar}[S^z, \boldsymbol{S}]e^{-i\omega S^z t/\hbar}\,. \tag{7.18}$$

The z-component of this equation of motion is simple, because $[S^z, S^z] = 0$, so we have

$$\frac{\mathrm{d}S^z}{\mathrm{d}t} = 0\,. \tag{7.19}$$

The x-component is found using the fact that $[S^z, S^x] = i\hbar S^y$, which gives

$$\frac{\mathrm{d}S^x}{\mathrm{d}t} = \frac{i\omega}{\hbar}e^{i\omega S^z t/\hbar}i\hbar S^y e^{-i\omega S^z t/\hbar} = -\omega S^y(t)\,. \tag{7.20}$$

Similarly, using $[S^z, S^y] = -i\hbar S^x$, we find

$$\frac{\mathrm{d}S^y}{\mathrm{d}t} = \omega S^x(t)\,. \tag{7.21}$$

We can write these three expression compactly in vector notation as

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = -\omega \boldsymbol{S} \times \hat{\boldsymbol{z}} \,. \tag{7.22}$$

This is the same equation as for classical spins, but the interpretation is entirely different because $\boldsymbol{S}$ is now an operator. If we take the expectation value of each side of this expression, we will get the same answers that we found in the Schrödinger picture.

7.2 Particle in a Potential

We have said, on general grounds, that time evolution is given by a unitary operator. Furthermore, we have said that the infinitesimal form of this operator corresponds to a Hermitian operator, which we have called the Hamiltonian. However, we must ask: how do we specify a quantum mechanical system? There are two main ingredients we need: we have to first specify the Hilbert space that the states live in, and then we have to specify the Hamiltonian. Either of these is insufficient without the other.

For a particle in a potential, the Hilbert space is the space of square-integrable functions (modulo magnitude and phase). In order to specify the Hamiltonian, we will simply take the classical Hamiltonian and replace the variables x and p by the corresponding operators x and p. We must be careful, because although the regular variables x and p commute, the corresponding operators do not.

Suppose we have the Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \,. \tag{7.23}$$

How do we proceed? In the Schrödinger picture, we first find the energy eigenkets $|j\rangle$ and eigenvalues E_j, which satisfy

$$H|j\rangle = E_j|j\rangle \,. \tag{7.24}$$

Once we have found these eigenkets, we can expand an arbitrary state as

$$|\psi\rangle = \sum_j c_j|j\rangle \,. \tag{7.25}$$

Time evolution is carried out as

$$|\psi(t)\rangle = \sum_j c_j e^{-iE_jt/\hbar}|j\rangle \,. \tag{7.26}$$

In principle, this is a complete procedure to determine the dynamics of any quantum system. However, in practice this can often be very difficult.

In the Heisenberg picture, we simply care about the operators,

$$\begin{aligned} x_{\mathrm{H}}(t) &= e^{iHt/\hbar} x e^{-iHt/\hbar} \,, \\ p_{\mathrm{H}}(t) &= e^{iHt/\hbar} p e^{-iHt/\hbar} \,. \end{aligned} \tag{7.27}$$

Using the Heisenberg picture equation of motion, we see that the operator x_H evolves according to

$$\frac{\mathrm{d}x_{\mathrm{H}}}{\mathrm{d}t} = \frac{1}{i\hbar}[x_{\mathrm{H}}, H] = \frac{1}{i\hbar}\left[x_{\mathrm{H}}, \frac{p_{\mathrm{H}}^2}{2m} + V(x_{\mathrm{H}})\right] = \frac{1}{i\hbar}\left[x_{\mathrm{H}}, \frac{p_{\mathrm{H}}^2}{2m}\right] . \tag{7.28}$$

From this point forward, we will drop the subscript H on the Heisenberg picture operators. Using a commutator identity, we then have

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{1}{i\hbar}\left[x, \frac{p^2}{2m}\right] \\ &= \frac{1}{i\hbar}\left(\left[x, \frac{p}{2m}\right]p + p\left[x, \frac{p}{2m}\right]\right) \\ &= \frac{p}{m}\,.\end{aligned} \tag{7.29}$$

This is the expected result, but it is now an operator equation. Similarly, we find

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{1}{i\hbar}[p, V(x)] \\ &= \frac{1}{i\hbar}\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}(V(x)\cdot) + V(x)i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\cdot\right) \\ &= -\frac{\mathrm{d}V}{\mathrm{d}x}\,.\end{aligned} \tag{7.30}$$

Here, we have made clear the action of the derivative operator by using $\cdot$ to denote an arbitrary position-space wavefunction on which these operators could act. Once again, this resembles the classical expression, but is now an operator equation.

If we take the expectation values of Eqs. (7.29) and (7.30), we find

$$\begin{aligned}\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} &= \frac{\langle p\rangle}{m}\,, \\ \frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} &= -\left\langle\frac{\mathrm{d}V}{\mathrm{d}x}\right\rangle.\end{aligned} \tag{7.31}$$

This result is called *Ehrenfest's theorem*. It is important, when using the second equation, to remember to differentiate the potential first before taking the expectation value, because reversing the order of these operators will often change the result.

7.2.1 Example: Charged Particle in a Uniform Electric Field

Consider a charged particle in a uniform electric field, which has the Hamiltonian

$$H = \frac{p^2}{2m} - qE(t)x\,. \tag{7.32}$$

In the Schrödinger picture, this is a messy problem to solve. In the Heisenberg picture, however, the problem is not difficult at all. Using Eq. (7.30), we have

$$\frac{\mathrm{d}p}{\mathrm{d}t} = qE(t)\,. \tag{7.33}$$

This has solution

$$p(t) = p(0) + \int_0^t \mathrm{d}t'\, qE\big(t'\big)\,. \tag{7.34}$$

Similarly, using Eq. (7.29), we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p(t)}{m}\,, \tag{7.35}$$

which yields

$$x(t) - x(0) = \frac{tp(0)}{m} + \frac{q}{m}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\, E\big(t''\big)\,. \tag{7.36}$$

These are the same results we find classically. If $E(t) = E$ is independent of time, then we have

$$p(t) = p(0) + qEt\,, \quad x(t) = x(0) + \frac{p(0)t}{m} + \frac{q}{2m}Et^2\,. \tag{7.37}$$

7.2.2 Example: Simple Harmonic Oscillator

Recall the simple harmonic oscillator Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\,. \tag{7.38}$$

We are likely all familiar with the approach in the Schrödinger picture. In the Heisenberg picture, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p}{m}\,, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -m\omega^2 x\,. \tag{7.39}$$

Solving these equations gives

$$\begin{aligned} x(t) &= x(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t)\,, \\ p(t) &= -m\omega x(0)\sin(\omega t) + p(0)\cos(\omega t)\,. \end{aligned} \tag{7.40}$$

Finding these equations in the Schrödinger picture is messy, though it can be done; in the Heisenberg picture, the result was immediate.

Keep in mind that these are operator equations. If, for example, we square the operators $x(t)$ or $p(t)$, we must be careful with the order of $x(0)$ and $p(0)$ in the cross terms, as these operators do not commute.