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8.323 Relativistic Quantum Field Theory I
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8.323: Relativistic Quantum Field Theory I

Prof. Alan Guth

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LECTURE NOTES 6
PATH INTEGRALS, GREEN'S FUNCTIONS,
AND GENERATING FUNCTIONALS

In these notes we will extend the path integral methods discussed in Lecture Notes 5 to describe Green's functions, which we define to be ground state expectation values of the time-ordered product of Heisenberg operators. For the case of a nonrelativistic particle moving in one dimension, discussed in Lecture Notes 5, the Green's functions can be written as

$$\begin{aligned} G(t_N, \dots, t_1) &\equiv \langle 0 | T \{ x(t_N) x(t_{N-1}) \dots x(t_1) \} | 0 \rangle \\ &= \langle 0 | x(t_N) x(t_{N-1}) \dots x(t_1) | 0 \rangle , \end{aligned} \tag{6.1}$$

where $|0\rangle$ denotes the ground state, and the second line assumes that we have labeled the time arguments so that they are time-ordered, in the sense that

$$t_N \geq t_{N-1} \geq \dots \geq t_1 . \tag{6.2}$$

In the quantum field theory, the Green's functions will be defined analogously by

$$\begin{aligned} G(x_N, \dots, x_1) &\equiv \langle 0 | T \{ \phi(x_N) \dots \phi(x_1) \} | 0 \rangle , \\ &= \langle 0 | \phi(x_N) \dots \phi(x_1) | 0 \rangle , \end{aligned} \tag{6.3}$$

where $|0\rangle$ denotes the vacuum state, and again the second line assumes that the time arguments are time-ordered. In the nonrelativistic quantum mechanics example of Eq. (6.1), the Green's functions are not quantities that are particularly interesting, so they are usually never mentioned in a course in quantum mechanics. We will soon see, however, that the quantum field theory Green's functions of Eq. (6.3) are **very** interesting. In particular, the entire formalism for calculating scattering cross sections and decay rates will be based upon relating these quantities to the Green's functions. In addition to showing how to express these Green's functions as path integrals, in these notes we will also see that one can define a generating functional $Z[J]$ in such a way that the Green's functions can be expressed simply in terms of the functional derivatives of the generating functional.

GREEN'S FUNCTIONS:

To begin, we recall that in Lecture Notes 5 we learned to express the evolution operator of quantum mechanics as a path integral:

$$U_{fi} = \int_{x(0)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t) e^{\frac{i}{\hbar} S[x(t)]} , \quad (6.4)$$

where

$$\begin{aligned} S[x(t)] &= \int_0^{t_f} dt L(x, \dot{x}) \quad (L = \text{Lagrangian}) \\ &= \int_0^{t_f} dt \left\{ \frac{1}{2} m \dot{x}^2 - V(x) \right\} . \end{aligned} \quad (6.5)$$

We also know that the Heisenberg field operators appearing in Eq. (6.1) can be written as

$$x(t) = e^{iHt} x(0) e^{-iHt} , \quad (6.6)$$

where $x(0) \equiv x_S$ is the Schrödinger representation position operator. Eq. (6.1) can then be rewritten as

$$G(t_N, \dots, t_1) = \left\langle 0 \left| e^{iHt_N} x_S e^{-iH(t_N-t_{N-1})} x_S \dots e^{-iH(t_2-t_1)} x_S e^{-iHt_1} \right| 0 \right\rangle . \quad (6.7)$$

To express this quantity as a path integral, we can insert at each operator x_S a complete set of states in the position representation, using

$$\text{Identity Operator} = \int_{-\infty}^{\infty} dx |x\rangle \langle x| , \quad (6.8)$$

which can be multiplied by x_S to give

$$x_S = \int_{-\infty}^{\infty} dx x_S |x\rangle \langle x| = \int_{-\infty}^{\infty} dx |x\rangle x \langle x| . \quad (6.9)$$

Then

$$\begin{aligned} G(t_N, \dots, t_1) &= \int_{-\infty}^{\infty} dx_1 \dots dx_N \\ &\quad \langle 0 | e^{iHt_N} |x_N\rangle x_N \langle x_N | e^{-iH(t_N-t_{N-1})} |x_{N-1}\rangle x_{N-1} \dots \\ &\quad \times x_2 \langle x_2 | e^{-iH(t_2-t_1)} |x_1\rangle x_1 \langle x_1 | e^{-iHt_1} |0\rangle . \end{aligned} \quad (6.10)$$

The matrix elements in this expression can all be written as path integrals, except for the ground state matrix elements on the two ends. Even these matrix elements can be treated by path integral methods, however, by noting that the ground state is defined in terms of the Hamiltonian, and path integrals can be used to construct matrix elements of exponentials of the Hamiltonian. An arbitrary state, such as the state of definite position $|x_0\rangle$ for some constant x_0 , can always be expanded in energy eigenstates:

$$|x_0\rangle = \sum_n |\psi_n\rangle \langle \psi_n | x_0 \rangle . \quad (6.11)$$

We can isolate the ground state contribution to this equation by multiplying by sides by $e^{-\xi H}$, where ξ is some real number:

$$e^{-\xi H} |x_0\rangle = \sum_n e^{-\xi E_n} |\psi_n\rangle \langle \psi_n | x_0 \rangle , \quad (6.12)$$

where on the right the operator H has been replaced by its eigenvalue E_n , where $H |\psi_n\rangle = E_n |\psi_n\rangle$. As ξ becomes large, the ground state contribution to the right-hand side will be less suppressed than any other state. We can compensate for this suppression by multiplying by $e^{\xi E_0}$, where E_0 is the energy of the ground state $|\psi_0\rangle \equiv |0\rangle$. Thus,

$$\lim_{\xi \rightarrow \infty} e^{\xi E_0} e^{-\xi H} |x_0\rangle = |\psi_0\rangle \langle \psi_0 | x_0 \rangle . \quad (6.13)$$

For our path integral it will be more convenient to describe the real exponential in the above equation as a “small” correction to a much larger imaginary exponential. We introduce a variable T , with units of time, taking the limit as T approaches infinity times $(1 - i\epsilon)$, where ϵ is a small positive constant. At the end we will take the limit $\epsilon \rightarrow 0$, but only after the infinite limit is carried out, so that $\infty \cdot \epsilon = \infty$. If we assume that $\langle \psi_0 | x_0 \rangle \neq 0$, meaning that we have not chosen x_0 to be a point where the ground state wave function vanishes, then we can divide both sides of Eq. (6.13) by this quantity, obtaining

$$|\psi_0\rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} e^{-iHT} |x_0\rangle \frac{e^{iE_0 T}}{\langle \psi_0 | x_0 \rangle} . \quad (6.14)$$

For the bra vector, we can use the analogous relation

$$\langle \psi_0 | = \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{e^{iE_0 T}}{\langle x_0 | \psi_0 \rangle} \langle x_0 | e^{-iHT} . \quad (6.15)$$

Note that Eq. (6.15) was not obtained by simply taking the adjoint of Eq. (6.14), because the adjoint equation would involve T^* instead of T , which would not be useful for our current purposes.

If we use Eqs. (6.14) and (6.15) to replace both ground state matrix elements in Eq. (6.10), we obtain

$$\begin{aligned}
G(t_N, \dots, t_1) = & \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{e^{2iE_0T}}{\langle x_0 | \psi_0 \rangle \langle \psi_0 | x_0 \rangle} \int_{-\infty}^{\infty} dx_1 \dots dx_N \\
& \langle x_0 | e^{-iH(T-t_N)} | x_N \rangle x_N \langle x_N | e^{-iH(t_N-t_{N-1})} | x_{N-1} \rangle x_{N-1} \dots \\
& \times x_2 \langle x_2 | e^{-iH(t_2-t_1)} | x_1 \rangle x_1 \langle x_1 | e^{-iH(T+t_1)} | x_0 \rangle .
\end{aligned} \tag{6.16}$$

Thus, we see that the path integral is so smart that it can even calculate the ground state wave function for us. The last matrix element in Eq. (6.16) can also be written as

$$\langle x_1 | e^{-iH(t_1-(-T))} | x_0 \rangle ,$$

which can be described as the evolution operator from time $-T$ to time t_1 .

If each matrix element in Eq. (6.16) is expressed as a path integral by using Eq. (6.4), we find

$$\begin{aligned}
G(t_N, \dots, t_1) = & \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{e^{2iE_0T}}{\langle x_0 | \psi_0 \rangle \langle \psi_0 | x_0 \rangle} \int_{-\infty}^{\infty} dx_1 \dots dx_N \\
& \int_{x(T)=x_0}^{x(t_N)=x_N} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]} x_N \int_{x(t_N)=x_N}^{x(t_{N-1})=x_{N-1}} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]} x_{N-1} \dots \\
& \times x_2 \int_{x(t_2)=x_2}^{x(t_1)=x_1} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]} x_1 \int_{x(t_1)=x_1}^{x(-T)=x_0} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]} .
\end{aligned} \tag{6.17}$$

We see that for any t which is not equal to one of the set $\{t_1, \dots, t_N\}$, the quantity $x(t)$ appears as one of the variables of integration in one of the path integrals. For each t_i in the set $\{t_1, \dots, t_N\}$, $x(t_i)$ is required by the limits of integration to be x_i , which is then integrated from $-\infty$ to ∞ . Thus, $x(t)$ is actually a variable of integration for all values of t , provided that we recognize that $x(t_i) \equiv x_i$. All the path integrals can then be combined into one path integral from time $-T$ to T , so Eq. (6.17) simplifies enormously:

$$\begin{aligned}
G(t_N, \dots, t_1) = & \\
& \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{e^{2iE_0T}}{\langle x_0 | \psi_0 \rangle \langle \psi_0 | x_0 \rangle} \int_{x(-T)=x_0}^{x(T)=x_0} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]} x(t_N) \dots x(t_1) .
\end{aligned} \tag{6.18}$$

The complicated factor in front of the path integral can be cancelled if we divide the expression by

$$\langle 0 | e^{-2iHT} | 0 \rangle = \frac{e^{2iE_0T}}{\langle x_0 | \psi_0 \rangle \langle \psi_0 | x_0 \rangle} \int_{x(-T)=x_0}^{x(T)=x_0} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]}, \quad (6.19)$$

which gives finally

$$G(t_N, \dots, t_1) = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int_{x(-T)=x_0}^{x(T)=x_0} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]} x(t_N) \dots x(t_1)}{\int_{x(-T)=x_0}^{x(T)=x_0} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]}}. \quad (6.20)$$

In defining the Green's functions, we made the explicit choice in Eq. (6.3) that we would use time-ordered products. Since the operators $x(t_i)$ do not in general commute, we presumably would have found a different answer if we had used a different ordering. Nonetheless, in our final result (6.20), the ordering is not apparent. The product $x(t_N) \dots x(t_1)$ is just a product of c-numbers in the integrand, so the product would have the same value if the factors were written in any order. Thus, we see that the path integral naturally picks out the time-ordered product. It will turn out, however, that the time-ordered product is exactly what we will need to calculate cross sections, so there is a perfect fit between the technique and the needed output.

Operator products which are not time-ordered are still well-defined, however, so there ought to be some path integral method that would allow one to calculate them if one wanted to. If you read over the previous derivation and think about what would be different if the operators were not time-ordered, you would find that nothing would change until the step that turns Eq. (6.16) into Eq. (6.17). For time-ordered operators, all the time arguments appearing in the exponents of Eq. (6.16) are positive semidefinite, since $t_{n+1} \geq t_n$. Thus, each matrix element is an evolution operator that evolves forward in time. If the operators were not time-ordered, then some of the time arguments would be negative, corresponding to an evolution operator backwards in time. Such evolution operators can be expressed as path integrals, too, but the sum is over paths that go backwards in time. When the right-hand side of Eq. (6.17) is combined into a single path integral, as in Eq. (6.18), the paths $x(t)$ would have to zigzag in time, sometimes going forward and sometimes going backwards, to reproduce the matrix elements in Eq. (6.16).

GENERATING FUNCTIONALS:

Eq. (6.20) can be conveniently rewritten by the use of a generating functional, defined by

$$Z[J(t)] = \lim_{T \rightarrow \infty(1-i\epsilon)} \int_{-T}^T \mathcal{D}x(t) e^{\frac{i}{\hbar} \int_{-T}^T dt [L(x, \dot{x}) + J(t)x(t)]} . \quad (6.21)$$

It will then be possible to express the Green's functions as derivatives of the generating functional.

The derivative of a functional is called a functional derivative, as you might guess, but the definition is slightly indirect. Crudely speaking the functional $Z[J(t)]$ is a function of an infinite number of arguments, $J(t)$ for each value of t , so the derivative should look something like a partial derivative. Partial derivatives are defined in terms of the variation of the function when one argument is varied with the other arguments fixed, but that will not work for $Z[J(t)]$. If we vary $J(t)$ for one value of t only, $Z[J(t)]$ will not change at all, since the one point would have measure zero in the integration of Eq. (6.21). So, the functional derivative is defined by first thinking about how a function of many variables changes when all of its variables are changed by a small amount. If a function of N variables is denoted by $F(z_1, \dots, z_N)$, then its first order Taylor expansion can be written

$$F(z_1 + \Delta z_1, \dots, z_N + \Delta z_N) = F(z_1, \dots, z_N) + \sum_{j=1}^N \frac{\partial F}{\partial z_j} \Delta z_j + \mathcal{O}(\Delta z^2) . \quad (6.22)$$

Eq. (6.22) could be used as the definition of $\partial F / \partial z_j$, which would be equivalent to the usual definition. The functional derivative $\delta Z / \delta J(t)$ is defined in analogy to Eq. (6.22):

$$Z[J(t) + \Delta J(t)] \equiv Z[J(t)] + \int dt' \frac{\delta Z}{\delta J(t')} \Delta J(t') + \mathcal{O}(\Delta J^2) . \quad (6.23)$$

To calculate the functional derivative of Eq. (6.21), we write

$$\begin{aligned} Z[J(t) + \Delta J(t)] &= \int_{-T}^T \mathcal{D}x(t) e^{\frac{i}{\hbar} \int_{-T}^T dt [L(x, \dot{x}) + [J(t) + \Delta J(t)]x(t)]} \\ &= \int_{-T}^T \mathcal{D}x(t) e^{\frac{i}{\hbar} \int_{-T}^T dt [L(x, \dot{x}) + J(t)x(t)]} \left[1 + \frac{i}{\hbar} \int_{-T}^T dt' \Delta J(t') x(t') \right] \\ &= Z[J(t)] + \frac{i}{\hbar} \int_{-T}^T dt' \Delta J(t') \int_{-T}^T \mathcal{D}x(t) e^{\frac{i}{\hbar} \int_{-T}^T dt [L(x, \dot{x}) + J(t)x(t)]} x(t') . \end{aligned} \quad (6.24)$$

Comparing Eqs. (6.23) and (6.24), one sees that

$$\frac{\delta Z[J(t)]}{\delta J(t')} = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{i}{\hbar} \int_{-T}^T \mathcal{D}x(t) e^{i \int_{-T}^T dt [L(x, \dot{x}) + J(t)x(t)]} x(t') . \quad (6.25)$$

Thus, referring to Eq. (6.20), one sees that

$$\frac{1}{Z[J(t)]} \frac{\delta Z[J(t)]}{\delta J(t)} \Big|_{J=0} = \frac{i}{\hbar} G(t) . \quad (6.26)$$

Since Eq. (6.25) implies that the functional differentiation $\delta/\delta J(t')$ brings down a factor of $\frac{i}{\hbar}x(t')$ in the integrand, it is easy to see that successive functional differentiations would bring down successive factors of $\frac{i}{\hbar}x(t)$. Thus,

$$G(x_N, \dots, x_1) \equiv \langle 0 | T \{ \phi(x_N) \dots \phi(x_1) \} | 0 \rangle ,$$

$$= \boxed{\frac{(-i\hbar)^N}{Z[J(t)]} \frac{\delta^N Z[J(t)]}{\delta J(t_1) \dots \delta J(t_N)} \Big|_{J=0}} . \quad (6.27)$$