Problems 1

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1. Aharonov-Bohm effect

   Consider the vector potential defined by

   $$A_x = \frac{-y}{x^2 + y^2} \tag{1}$$

   $$A_y = \frac{x}{x^2 + y^2} \tag{2}$$

   $$A_z = 0 \tag{3}$$

   a. Show that the curl of $\vec{A}$ is zero, apart possibly from the z-axis where it is ill-defined.

   b. Use Stoke's theorem to define the curl on the z-axis.

   Since $\vec{A}_i = \partial_i \phi$, where ϕ is the angle in cylindrical coordinates, you have now derived the charming formula

   $$\epsilon_{zij} \partial_i \partial_j \phi = 2\pi \delta(x)\delta(y). \tag{4}$$

   Can you think of a simple way to realize this set-up physically?

   c. Now suppose there is a repulsive barrier around the z-axis, so we don't have to worry about the singular nature of the vector potential there. Write down the Hamiltonian for a particle of charge q. Show that if the wave-function depends on the angle as $e^{in\phi}$, then the repulsive centrifugal barrier is proportional to $(n+q)^2$. If the charge q is not an integer[1] the spectrum will have changed, even though the particle never sees a non-zero field strength. This is the Aharonov-Bohm effect, which dramatically illustrates the unavoidable significance of the vector potential in quantum mechanics.

   Effectively, the particle has acquired fractional angular momentum.

   d. If you have been putting $\hbar = 1$, now restore it. Discuss in two sentences what is classical and what is quantum mechanical in the Aharonov-Bohm effect.

   e. [This part is Profound.] The vector potential appears to be axially symmetric. How is this consistent with its perturbing the angular momentum?

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   [1]If you prefer unit charge, let the vector potential be q times what we originally assumed.

2. Theory of the 3-index antisymmetric tensor field

   Consider a 3-index antisymmetric tensor field $A_{\mu\nu\rho}$ with the gauge symmetry

   $$A_{\mu\nu\rho} \to A_{\mu\nu\rho} + \partial_\mu \Lambda_{\nu\rho} \tag{5}$$

   where, in the last term, antisymmetrizaton over the indices is assumed.

   a. By analogy with electromagnetism, construct an invariant field-strength $F_{\mu\nu\rho\sigma}$ and a kinetic energy term. What is the free equation of motion (analogue of the Maxwell equations)? How many degrees of freedom does this field have?

   b. We saw how to couple the electromagnetic field to paticles in a gauge-invariant way by adding a world-line term $\int A_\mu dx^\mu$. What is the natural kind of object for $A_{\mu\nu\rho}$ to couple to? What does it take for this to be gauge-invariant? (If you don't have a comfortable grip on differential geometry, do a special case, analogous to a straight world-line.)

   c. What is the equation of motion for F now?

   d. There is a natural vector-field $B^\alpha = \epsilon^{\alpha\mu\nu\rho} A_{\mu\nu\rho}$ associated with the three-index tensor field. What are the salient features of B?

3. Spin and statistics with free fields

   a. Quantize the free scalar field using anticommutators (Fermi statistics). Show that fields at spacelike separations do not commute.

   b. Quantize the free Dirac equation using commutators (Bose statistics). Show that the bottom falls out of the Hamiltonian.

4. Some non-abelian group theory

   The theory of non-abelian groups and their representations is a big subject in itself, especially if explicit algorithms for classifying invariants and labelling and reducing representations are required. Fortunately in the Standard Model we need only to deal with the groups $SU(2)$ and $SU(3)$, and for these we can do everything we need using *ad hoc* "bare hands" methods. If any of the specific group-theoretic manipulations used in class are completely unfamiliar, you should ask me, but it might also be a good idea to review the basic ideas, to make sure you're comfortable with them. As a reference, see for example M. Hamermesh, *Group Theory and its Application to Physical Problems*[2]. Here are some simple exercises to focus your attention.

   a. The infinitesimal versions of group elements near the identity define the Lie algebra of the group. Thus we write

   $$\Omega^a_b = \delta^a_b + i\omega^a_b \tag{6}$$

   with ω small. Show that the condition that Ω is unitary translates (dropping quadratic terms!) into the condition that ω is Hermitean, and that the condition that Ω has unit determinant translates into the condition that ω has zero trace.

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[2]This is a really excellent book and reference that every theoretical physicist should own, especially since it is available in a cheap Dover edition.

Since these conditions are linear, the Lie algebra forms a vector space. Find nice bases for the $SU(2)$ and $SU(3)$ cases. (A nice basis is one for which $\mathrm{Tr}\,\omega^a\omega^b = \frac{1}{2}\delta^{ab}$.) What are their dimensions? Given a basis, we define the structure constants by

$$[\omega^a, \omega^b] = if^{ab}_c \omega^c. \tag{7}$$

Compute the structure constants for your bases.

b. Show that the group of 2×2 unitary transformations can be parametrized in the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \tag{8}$$

with α and β being complex numbers subject to $|\alpha|^2 + |\beta|^2 = 1$. (So what is the topology of $SU(2)$?)

Work out a similar parametrization for $SU(3)$. [Hint: The first row is an abitrary vector of unit magnitude; the second is orthogonal to it; the third is proportional to the cross-product.] How many real parameters are required? Optional: What does this construction suggest about the topology of $SU(3)$?

c. Upper-index (covariant) vectors transform as columns that get multiplied on the left; lower-index (contravariant) vectors transform as rows that get multiplied on the right by the inverse matrix. Thus

$$\begin{aligned} (\phi')^\mu &= \Omega^\mu_\nu \phi^\nu; \\ (\psi')_\mu &= (\Omega^{-1})^\sigma_\mu \psi_\sigma. \end{aligned} \tag{9}$$

Show that the complex conjugate of a covariant vector transforms like a contravariant vector.

¿From these we can construct tensors, which transform like products of vectors. Convince yourself that we can apply symmetry restrictions on tensors (e.g., requiring them to be symmetric or antisymmetric) and still have representations of the groups.

Show by explicit calculation that the Kronecker delta δ^a_b and the antisymmetric symbol ϵ_{ab} are invariant under $SU(2)$ transformations. Use this to argue that any tensor can be decomposed into a sum of symmetric tensors with up indices only. What are the dimensions of these representations? How does the group transformation $\Omega = e^{i\frac{\tau 3}{2}}$ act on these? Does this ring any bells?

d. We define Lie-algebra representations in general by matrices that satisfy the commutation relations

$$[\rho^a, \rho^b] = if^{ab}_c \rho^c \tag{10}$$

where the f^{ab}_c are the structure constants of the group. Convince yourself that this "works", in the sense that the representation matrices you get by taking infinitesimal versions of how the group elements work on the tensor product obey this rule.

For physics purposes we are often interested in the so-called Casimir number $C_2(\rho)$ and trace $\mathrm{Tr}(\rho)$ associated with a representation. These are defined by

$$C_2(\rho)\delta^\mu_\nu = (\rho^a)^\mu_\sigma (\rho^a)^\sigma_\nu, \tag{11}$$

with summation over a and σ assumed, and by

$$T(\rho)\delta^{ab} = \text{Tr}\rho^a\rho^b \tag{12}$$

using a nice basis[3]. Thus T(fundamental) for the fundamental vector representation is, by definition, $\frac{1}{2}$.

Prove the general relationship

$$d_\rho C_2(\rho) = d_G T(\rho), \tag{13}$$

where d_ρ is the dimension of the representation and d_G is the dimension of the group.

Evaluate $C_2(\rho)$ for the representations of $SU(2)$ discussed above.

Evaluate $C_2(\rho)$ for the fundamental vector **3** representation of $SU(3)$, and for the following:

- the **6** representation, using a two-index symmetric tensor
- the **10** representation, using a three-index symmetric
- the adjoint **8** representation, using a traceless tensor with one upper and one lower index

[Hint: Compute T by picking a convenient Lie algebra element.]

For a grand finale, compute $C_2(\mathbf{27})$ for the representation of $SU(3)$ that has two symmetric upper indices, two symmetric lower indices, and vanishing trace.

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[3]These "definitions" assume that the forms on the two sides match. That will be true for irreducible representations.