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**PROFESSOR:** We looked at the macro state, where the volume of the box, the temperature were given, as well as the chemical potential. So rather than looking at a fixed number of particles, we were looking at the grand canonical prescription where the chemical potential was specified. We said that when we have a system of non-interacting particles, we can construct a many bodied state based on occupation of single particle states such that the action of the Hamiltonian on this will give me a sum over all single particle states, the occupation number of that single particle state times the energy of that single particle state, and then the state back.

We said that in quantum mechanics, we have to distinguish between bosons and fermions, and in the occupation number prescription. For the case of fermions, corresponding  $\theta_{\nu} - 1$  and  $k$  was 01. For the case of bosons, corresponding  $\theta_{\nu} + 1$  and  $k$  equals [INAUDIBLE].

We said that within this planned canonical prescription, these occupation numbers were independently given. The probabilities were independently distributed according to exponential rules such that when we calculated the average,  $\langle n_k \rangle$ . We commented on the statistics what are the very simple forms, which was  $1$  over [INAUDIBLE] into the  $\beta \epsilon_k - \mu$ , where  $z$  was constructed from the temperature and the temporal potential as [INAUDIBLE] to the  $\beta \mu$ .

Now, for the case that we are interested, for this gas that is in a box of volume  $v$ , the one particle states were characterized by [INAUDIBLE] waves and their energies were  $\frac{\hbar^2 k^2}{2m}$ , basically just the kinetic energy. And then at various stages, we have to perform sums over  $k$ 's, and for a box of volume  $v$ , the sum over  $k$  was replaced in the limit of a large box with an integral over  $k$  times the density of states, which was  $v$  divided by  $2\pi^3$ . And we said that again, we

recognized that quantum particles have another characteristic, which is their spin, and there is a degeneracy factor associated with that that we can just stick over here as if we had [INAUDIBLE] copies corresponding to the different components of the spin.

Now, once we put this form of  $\epsilon_k$  over here, then we can calculate what the mean number of particles is in the grand canonical ensemble as sum over  $k$ , the average  $n_k$ , given the appropriate statistics. Of course, this is a quantity that is extensive. We then construct a density, which is intensive, simply by dividing by  $v$ , which gets rid of that factor of  $v$  over here, and we found that the answer could be written as  $g$  over  $\lambda^3$ , again, coming from this  $g$ ,  $\lambda^3$  from the change of variables, where we defined  $\lambda = h / \sqrt{2\pi m k_B T}$ .

And then a function that we indicated by  $f_{3/2}$ , depending on the statistics of  $z$ . And this was a member of our class of functions we defined in general,  $f_m$  of  $z$ , to be  $1 / (m-1)!$ , integral 0 to infinity,  $e^{-x} x^{m-1} z^{-x}$ ,  $dx$ . So for each  $m$ , we have a function of  $z$ . In fact, we have two functions, depending on whether we are dealing with the fermionic or the bosonic variety.

We saw that these functions we could expand for small  $z$  as  $z + \eta z^2 / 2 + \dots$ . It's an alternating series for fermions. All terms are positive for bosons,  $z^3 / 3! + \eta z^4 / 4! + \dots$ , and so forth. And we also noted that these functions have a nice property in that if you were to take a derivative with respect to  $z$  and multiply by  $z$ , you get the function with one lower index. So this is, in fact, the same thing as  $z \frac{d}{dz} f_m = f_{m-1}$  [INAUDIBLE].

This just made the connection between the number of particles and the chemical potential that we need to use when we go to the grand canonical prescription, but we are interested in calculating various properties of this gas, such as the pressure. We found that the formula for the pressure was simply  $g / \lambda^3$ , very similar to what we had over here, except that, rather than having  $f_{3/2}$ , we had  $f_{5/2}$

of  $z$  to deal with, and if we were interested in energy, we could simply use  $3/2$ . Indeed, this was correct for both fermions and bosons, irrespective of the difficulties or the variations in pressure that you would have, depending on your bosons or fermions due to [INAUDIBLE] gases.

So what is the task here, if you are interested in getting a formula for pressure or the energy or the heat capacity? What should we do if we are interested in a gas that has a fixed number of particles or a fixed density? Clearly, the first stage is to calculate  $z$  as a function of density. What we can do is to solve the first equation graphically. We have to solve the equation  $f^{3/2} \cdot 8$  of  $z$  equals the combination  $m$  lambda cubed over  $g$ , which we call the degeneracy factor.

So if I tell you what the temperature is, you know lambda. If I tell you what the density is, you know the combination on the right hand side. There is this function. We have to plot this function and find the value of the argument of the function. That gives the value of  $m$  lambda cubed over  $g$  as the value of the function.

So graphically, what we have to do is to plot as a function of  $z$  these functions,  $f^{3/2}$ . I will generically plot  $f^m$  of  $z$ , but for the case that we are interested, we really need to look at  $m$  equals  $3/2$ . As we will see in problem set, if you were to solve the gas in  $d$  dimensions, this  $m$  rather than the  $3/2$  would be  $g$  over  $2$ . So pictorially, the same thing would work with slightly different forms of these functions in different dimensions.

How do these functions look like? Well, we see that initially, they start to be linear, but then, the case of the bosons and fermions, the next order term goes in different directions. And in particular, for the case of fermions, the quadratic term starts to bring you down, whereas for the case of bosons, the quadratic term is the opposite direction and will tend to make you large. This is what these functions look.

For the case of fermions, I have additional terms in the series. It's an alternating series. And then the question is, what happens to this at large values of  $z$ ? And what we found was that at large values, this function satisfies the asymptotic form that is provided by Sommerfeld formula, which states that  $f^m$  of  $z$ , for the case of

fermions, so this is  $\eta = -1$ , is  $\log z$  to the power of  $m$  divided by  $m$  factorial. Of course, we are interested in  $m$  goes to  $3/2$ .  $1 + \pi^2/6$ ,  $m - 1$ ,  $\log z$  squared, and higher order inverse powers of  $\log z$ , which one can compute, but for our purposes, these two terms are sufficient.

And if I'm interested in doing the inversion for fermions, what I need to do is to plot a line that corresponds to  $n \lambda^3 / g$  and find the intersection of that line, and that would give me the value of  $z$ . When I'm very close to the origin, I start with the linear behavior, and then I can systematically calculate the corrections to the linear behavior, which would have been  $z = n \lambda^3 / g$  in higher powers of  $n \lambda^3 / g$ . And we saw that by doing that, I can gradually construct a visual expansion that is appropriate for these gases.

I can do the same thing, of course, for the case of bosons also, but now I'm interested in the limiting form where the density goes higher and higher or the temperature goes lower and lower so that this horizontal line that I have drawn will go to larger and larger values, and I'm interested in the intersections that I have when  $z$  is large and this asymptotic formula is being satisfied. So what do I need to do for the case of degenerate fermions?

I have this quantity,  $n \lambda^3 / g$ , which I claim is much larger than 1, so I have to look for values of the function,  $f^{3/2}$  minus of  $z$  that are large, and those values are achieved when the argument is large, and then it takes the form  $\log z$  to the  $3/2$  divided by  $3/2$  factorial,  $1 + \pi^2/6$ .  $m$  in this case is  $3/2$ , so I have  $3/2$  times  $m - 1$ , which is  $1/2$ , divided by  $\log z$  over 2. And then I will have potentially higher order terms.

So I can invert this formula to find  $\log z$  as a function of density. And to lowest order, what I have is  $3/2$  factorial  $n \lambda^3 / g$  raised to the  $2/3$  power. If I were to ignore everything here, just arrange things this way, that's the formula that I would get. But then I have this correction, so I divide by this. I have  $1 + \pi^2/8$ ,  $1 / \log z$  squared. I have taken it to the other side, so I put the minus sign and raise again to the  $2/3$  power.

Now, the way that I have defined  $z$  up there, clearly  $\log z$  is proportional to  $\beta \mu$ . We can see that  $\lambda$  is proportional to  $\beta$  to the  $1/2$ , so  $\lambda^3$  is  $\beta$  to the  $3/2$ , which, when raised to the  $2/3$ , gives me a factor of  $\beta$ . So basically, the quantity that we have over here is of the order of  $\beta$ , and then we saw that the remainder, which depends on properties such as the mass of the gas,  $g$ , et cetera, which is independent of temperature, is a constant. That is the Fermi energy that we can compute by usual route of filling up a Fermi c. And the value of  $\epsilon_f$  is  $\hbar^2 k_f^2 / 2m$  times  $k_f^2$ , and the  $k_f$  was  $(6\pi^2 n)^{1/3}$ , so  $k_f^2$  would be something like this. So once we know the density of the fermion, its spin and its mass, we can figure out what this quantity is.

The zeroth order solution for the chemical potential is clearly this  $\epsilon_f$ . What do I have to do if I want to calculate the next correction? Well, what I can do is I can put the zeroth order solution for  $\log z$  in this expression, and then that will give me the first order correction. I have to raise this whole thing to the minus  $2/3$  power, so  $\pi^2 / 8$  becomes  $(1 - 2/3) \pi^2 / 8$ , which is  $\pi^2 / 12$ .  $1 / \log z$  is again  $kt / \mu$ , so its inverse is  $kt / \epsilon_f^2$ , and we expect that there will be higher order terms.

So what have I stated? If I really take this curve and try to solve for  $\mu$  as a function of temperature for a given density, I find that at very low temperatures, the value that I get is  $\epsilon_f$ , and then the value starts to get reduced as I go to higher temperatures and the initial fall is quadratic. Of course, the function will have higher and higher order corrections. It will not stay quadratic.

And I also know that at very large values, it essentially converges to a form that is  $-\mu / kt$  because down here, I know that this solution is  $z$  is simply  $n \lambda^3 / g$ . So at low densities and high temperatures, basically, I will have something that is almost linear with logarithmic corrections and negative, and so presumably, the function looks something like this.

Of course, I can always convert. We can say that what I see here is the combination

of  $kT$  over  $\epsilon_f$ . I can define  $\epsilon_f$  over  $kV$  to be something that I'll call  $t_f$ , and this correction I can write as  $t$  over  $t_f$  squared.

So basically, I have defined  $\epsilon_f$  to be some  $kV$  times a Fermi temperature. Just on the basis of dimensional argument, you would expect that the place where this zero is occurring is of the order of this  $t_f$ . I don't claim that it is exactly  $t_f$ . It will have some multiplicative factor, but it will be of the order of  $t_f$ , which can be related to the density, mass, and other properties of the gas by this formula. And if you look at a metal, such as copper, where the electrons are approximately described by this Fermi gas, this  $t_f$  is of the order of, say, 10 to the 4 degrees Kelvin. Yes?

**AUDIENCE:** So you were saying that at a high temperature, the chemical potential should go as minus  $kVt$  times  $\ln$  of  $n$  lambda cubed, yeah?

**PROFESSOR:** Yes.

**AUDIENCE:** Now, the lambda itself has temperature dependence, so how can you claim that it is behaving linearly?

**PROFESSOR:** I said almost linearly. As you say, the correct behavior is something like minus  $3/2 t \log t$ , so  $\log t$  changes the exponent to be slightly different from 1. This is not entirely linear. It has some curvature due to this logarithm. In fact, something that people sometimes get confused at is when you go to high temperatures,  $\beta$  goes to 0. So why doesn't  $z$  go to 1? Of course,  $\beta$  goes to 0 but  $\mu$  goes to infinity such that the product of 0 and infinity is actually still pretty large, partly because of this logarithm.

**AUDIENCE:** Could you just explain again why it's linear?

**PROFESSOR:** Down here it is linear, so I know that  $z$  is  $n$  lambda cubed over  $g$ .  $z$  is  $e$  to the  $\mu$  over  $kT$ . So  $\mu$  is  $kT \log$  of  $n$  lambda cubed over  $g$ . Remember lambda depends on temperature, so it is almost linear, except that really what is inside here has a temperature dependence. So it's really more like  $t \log t$ .

**AUDIENCE:** When  $z$  is small?

**PROFESSOR:** When  $z$  is small and only when  $z$  is small, and  $z$  small corresponds to being down here. Again, that's what I was saying. Down here,  $\beta$  goes to zero but  $\mu$  goes to minus infinity such that the product of 0 and minus infinity is still something that is large and negative because of this.

**AUDIENCE:** If the product is large, then isn't  $z$  large?

**PROFESSOR:** If the product is large and negative, then  $z$  is exponentially small. In fact, if I were to plot this function just without the higher order corrections, if I plot the function,  $kT \log$  of  $n \lambda^3 / g$ , you can see that it goes to 0 when the combination  $n \lambda^3 / g$  is 1 because  $\log$  of 1 is 0. So that function by itself, asymptotically this is what I have, it will come and do something like this, but it really is not something that we are dealing with. It's in some sense the classical version. So classically, your  $\mu$  is given by this formula, except that classically, you don't know what the volume factor is that you have to put there because you don't know what  $h$  is.

So that's the chemical potential, but the chemical potential is really, for our intents and purposes, a device that I need to put here in order to calculate the pressure, to calculate the energy, et cetera. So let's go to the next step. We are again in this limit.  $\beta p$  is  $g / \lambda^3 f^{5/2} \text{ plus } z$ .

That's going to give me  $g / \lambda^3$  in the limit where  $z$  is large-- we established that  $z$  is large--  $\log z$  to the  $5/2$  divided by  $5/2$  factorial using the Sommerfeld expansion,  $1 + \pi^2 / 6$ . Now my  $m$  is  $5/2$ .  $m - 1$  would be  $3/2$  divided by  $\log z$  squared in higher order terms.

**AUDIENCE:** Question.

**PROFESSOR:** Yes?

**AUDIENCE:** Why is that plus?

**PROFESSOR:** Because I made a mistake. Just to make my life and manipulations easier, I'll do the following. I write under this the formula for  $n$  being  $g / \lambda^3 f^{3/2} \text{ minus}$

z, which is  $g$  over  $\lambda^3 \log z$  to the  $3/2$  divided by  $3/2$  factorial,  $1$  plus-- I calculated the correction--  $\pi^2$  over  $6$  divided by various combination, which is  $\pi^2$  over  $8$ ,  $1$  over  $\log z$  squared.

Now what I will do is to divide these two equations. If I were to divide the numerator by the denominator, what do I get? The left hand side becomes  $\beta p$  over  $n$ . The right hand side, I can get rid of the ratio of  $\log z^{5/2}$  to  $\log z^{3/2}$  is just one factor of  $\log z$ .

What do I have when I divide  $5/2$  factorial by  $3/2$  factorial? I have  $5/2$ . This division allows me to get rid of some unwanted terms. And then what do I have here? This combination here is  $5\pi^2$  over  $8$  minus  $1\pi^2$  over  $8$ . Because of the division, I will put a minus sign here, so that becomes  $4\pi^2$  over  $8$ , which is  $\pi^2$  over  $2$ . So I will get  $1$  plus  $\pi^2$  over  $2$ , and at the order that we are dealing, we can replace  $1$  over  $\log z$  squared with  $kt$  over  $\epsilon f$  squared.

Now, for  $\log z$  over here, I can write  $\beta \mu$ , and for  $\mu$ , I can write the formula that I have up here. So you can see that once I do that, the  $\beta$ s cancel from the two sides of the equation, and what I get is that  $p$  is  $2/5$ , the inverse of  $5/2$ . The  $n$  I will take to the other side of the equation. To the lowest order  $\mu$  is  $\epsilon f$ , so I will put  $\epsilon f$ , but  $\mu$  is not exactly  $\epsilon f$ . It is  $1$  minus  $\pi^2$  over  $12$ ,  $kt$  over  $\epsilon f$  squared. And actually, there was this additional factor of  $1$  plus  $\pi^2$  over  $2$ ,  $kt$  over  $\epsilon f$  squared.

So what do we find? We find that to the zeroth order, pressure is related to the density but multiplied by  $\epsilon f$ , so that even at zero temperature, there is a finite value of pressure left. So we are used to ideal gases that are classical, and when we go to zero temperature, they start to stop moving around. Since they are not moving around, there is no pressure that is exerted, but for the Fermi gas, you cannot say that all of the particles are not moving around because that violates this exclusion of  $n$  being zero or one. You have to give particles more and more momentum so that they don't violate the condition that they should have different values of the occupation numbers, the Pauli exclusion.



And therefore, even at zero temperature, you will have particles that in the ground state are zipping around, and because they're moving around, they can hit the wall and exert pressure on it, and this is the value of the pressure. As you go to higher temperature, that pressure gets modified. We can see that balancing these two terms, there is an increase in pressure, which is  $1 + \frac{5\pi^2}{6} \frac{kt}{\epsilon_f}$ .

So you expect the pressure, if I plot it as a function of temperature, at zero temperature, it will have this constant value,  $p_f$ . As I put a higher temperature, there will be even more energy and more kinetic energy, and the pressure will rise. Eventually, at very high temperatures, I will regain the classical, rather pressure is proportional to temperature. That's the pressure of this ideal Fermi gas.

The energy. Well, the energy is simply what we said over here, always  $\frac{3}{2} p v$ . So it is  $\frac{3}{2}$ , the pressure over there multiplied by  $v$ . When I multiply the density by  $v$ , I will get the number of particles.  $\frac{3}{2}$  times  $\frac{5}{2}$  will give me  $\frac{3}{5}$ , so I will  $\frac{3}{5}$  and  $\epsilon_f$  at the lowest order.

Again, you've probably seen this already where you draw diagrams for filling up a Fermi c. Up to some particular  $k_f$ , you would say that all states are occupied. When  $\epsilon_f$  is the energy of this state that is sitting right at the edge of the Fermi c, and clearly, the energies of the particles vary all the way from zero up to  $\epsilon_f$ , so the average energy is going to be less than  $\epsilon_f$ . Dimensionally, it works out to be  $\frac{3}{5}$  of that. That's what that says. But the more important part is how it changes as a function of temperature. What you find is that it goes to  $1 + \frac{5\pi^2}{6} \frac{kt}{\epsilon_f}$  in high order terms.

**AUDIENCE:** Question. Should the 6 be 12?

**PROFESSOR:** 6 should be 12? Let's see. This is  $\frac{6}{12}$  minus  $\frac{1}{12}$  is indeed  $\frac{5}{12}$ . Thank you. So if I ask what's the heat capacity at constant volume, this is  $dE$  by  $dT$ , so just taking the derivative. The zeroth order term, of course, is irrelevant. It doesn't vary the temperature. It's this  $t$  squared that will give you the variations.

So what does it give me? I will have  $n \epsilon f$ . I have  $\frac{3}{5}$  times  $5 \pi^2$  over  $12$ . The derivative of this combination will give me  $2kb^2 t$  divided by  $\epsilon f^2$ . There will be, of course, higher order, but this combination you can see I can write as follows. It is extensive. It is proportional to  $n$ . There is one  $kb$  that I can take out, which is nice, because natural units of heat capacity, as we have emphasized, are  $kb$ .

This combination of numbers, I believe the  $5$  cancels.  $3$  times  $2$  divided by  $12$  gives me  $1/2$ , so I have  $\pi^2$  over  $2$ . And then I have the combination,  $kt$  over  $\epsilon f$ , which I can also write as  $t$  over  $t_f$ . So the heat capacity of the Fermi gas goes to  $0$  as  $T$  goes to zero temperature, in accord with the third law of thermodynamics, which we said is a consequence of quantum mechanics. So we have this result, that the proportionality is given by the inverse of  $t_f$ .

If I were to plot the heat capacity in units of  $nkb$  as a function of  $T$ , at very high temperatures, I will get the classical result, which is  $3/2$ , and then I will start to get corrections to that. Those corrections will gradually reduce this, so eventually, the function that I will get starts linearly and then gets matched to that. So the linear behavior is of the order of  $t$  over  $t_f$ , and presumably at some temperature that is of the order of  $t_f$ , you will switch to some classical behavior.

And as I said, if you look at metals, this  $t_f$  is very large. So when you look at the heat capacity of something like copper or some other metal, you find that there is a contribution from the electrons to the heat capacity that is linear.

Now, the reason for this linear behavior is also good to understand. This calculation that I did is necessary in order to establish what this precise factor of  $\pi^2$  over  $2$  or appropriate coefficient is, but the physical reason for the linearity you should be able to know.

Basically, we saw that the occupation numbers for the case of fermions have this form that is  $1$  over  $z$  inverse,  $e$  to the  $\epsilon$  plus  $1$ . If I plot the occupation numbers as a function of energy, for a case that is at zero temperature, you can see that the occupation number is  $1$  or  $0$ , so you have a picture that is like this. You switch from

1 to 0 at  $\epsilon_f$ . Oops, there was a  $\beta$  here that I forgot,  $\beta \epsilon_f$ .

When I go to finite temperature, this becomes fuzzy because this expectation value for the number, rather than being a one zero step function, becomes smoothed out, becomes a function such as this, and the width over which this smoothing takes place is of the order of  $kT$ . So what happens is that rather than having this short Fermi  $c$ , you will start having less particles here and more occupied particles over here. To do that, you have created energies going from here to here that you have stored that is of the order of this  $kT$ .

What fraction of the total number have you given this energy? Well, the fraction is given from here. It is the ratio of this to the entirety, and that ratio is  $t$ , or  $kT$ , divided by  $\epsilon_f$  is the fraction that has been given energy this. So the excitation energy that you have is this number, the total times this fraction times the energy that you have, which is  $kT$ . And if I take the derivative of this, I will get precisely this formula over here up to this factor of  $\pi^2/2$ , which I was very cavalier, but the scaling everything comes from this.

And to all intents and purposes, this is the characteristic of all Fermi systems, that as you go to low temperatures, there is a small fraction of electrons, if you like, that can be excited. That fraction goes to 0 at low temperatures as  $kT/\epsilon_f$ , and the typical energies that they have is of the order of  $kT$ . So essentially, many of the results for the degenerate Fermi gas you can obtain by taking classical results and substituting for the number of particles  $n kT/\epsilon_f$ .

You will see in the problem set that this kind of argument tells you something also about the magnetic response of a system of electrons, what is the susceptibility and why does the susceptibility saturate at low temperatures. It's just substituting for the number of particles this fraction,  $n kT/\epsilon_f$ , will give you lots of things.

This picture is also valid in all dimensions, whereas next, we'll be discussing the case of bosons, where the dependence is on temperature. There is an exponent here that determines which dimension you are or depends on dimensions. For the case of fermions, this linearity exists independently.

So now let's think about bosons and what's going to happen for bosons. For bosons, we saw that the series, rather than being alternating, all of the terms are adding up together so that the parabolic correction, rather than reducing the function for the case of bosons, will increase it. So for a particular value of density, we find that the  $z$  of fermions is larger than the  $z$  of bosons. So for a given density, what we will find is that the chemical potential, whereas for the case of fermions was deviating from the classical form by going up, for the case of bosons, we'll start to deviate from the classical form and go down.

Now, there is something, however, that the chemical potential cannot do for the case of bosons which it did it for the case of fermions, and that was to change sign. Why is that? Well, we have that over there, the occupation number for a boson is  $1$  over  $z$  inverse, which is  $e$  to the  $\beta \epsilon_k - \mu$ .  $z$  inverse is  $e$  to the  $-\beta \mu$ .

Now, this result was obtained from summing a geometric series, and the condition certainly is that this object, that is, the inverse of what was multiplying different terms in the series, has to be larger than  $1$ . Of course, it has to be larger than  $1$  so that I get a positive occupation number. If it was less than  $1$ , the geometric series was never convergent. Clearly, the negativity of the occupation number has no meaning.

So this being positive strictly immediately implies that  $\mu$  has to be less than  $\epsilon_k$  for all  $k$ . So it has to be certainly less than the minimum of  $\epsilon_k$  with respect to all  $k$ . And for the case that I'm looking at where my  $\epsilon_k$ 's are  $\hbar^2 k^2 / 2m$ , the lowest one corresponds to  $k$  equals  $0$ , so  $\mu$  has to be less than  $0$ . It can never go to the other side of this. Or alternatively,  $z$  has to be less than or approaches maybe  $1$ .

So there is certainly a barrier here that we are going to encounter for the case of bosons at  $z$  equals  $1$ . We should not go beyond that point. So let's see what's happening for  $z$  equals  $1$  to this function.

You see, my task remains the same. In order to find  $z$ , I have to solve graphically for the intersection of the curve that corresponds to  $f + 3/2$  and the curve that corresponds to the density  $n \lambda^3 / g$ . So what happens as I go to higher and higher values of  $n \lambda^3 / g$ ? You can see one possibility is that this curve just diverges as  $z$  approaches 1. Then for every value of  $n \lambda^3 / g$ , you will find some value of  $z$  that will gradually become closer and closer to 1.

But is that the scenario? For that, we need to know what the value of this  $f$  function is at  $z$  equals 1, so the limit  $f + 3/2$  as  $z$  goes to 1. How do you obtain that? Well, that is  $1 / (m - 1)!$ ,  $\int_0^\infty dx, x^{m-1}, e^{-x}$ , which is  $1 / (m - 1)!$ . I have to integrate this function.

The integrand, how does it look like?  $x^{m-1}, e^{-x}$ . Well, at large  $x$ , there is no problem. It goes to 0 exponentially. At small  $x$ , it goes to 0 as  $x^{m-2}$ . So basically, it's a curve such as this that I have to integrate.

And if the curve is like I have drawn it, there is no problem. I can find the integral underneath it and there actually is a finite value that has a name. It's called a zeta function, so this is some tabulated function that you can look at.

But you can see that if  $m$ , let's say, is 0, then it is  $dx / x^2$ . So the other possibility is that this is a function that diverges at the origin, which then may or may not be integrable. And we can see that this is finite and exists only for  $m$  that is larger than 1.

So in particular, we are interested in the case of  $m$  equals  $3/2$ . So then at  $z$  equals 1, I have a finite value. It is zeta of  $3/2$ . So basically, the function will come up to a finite value at  $z$  equals 1, which is this zeta of  $3/2$ , which you can look up in tables. It's 2.612.

Now, I tried to draw this curve as if it comes and hugs the vertical line tangentially, that is, with infinite slope, and that is the case. Why do I know that? Because the derivative of the function will be related to the function at one lower index. So the

derivative of  $f^{3/2}$  is really an  $f^{1/2}$ .  $f^{1/2}$  does not exist. It's a function that goes to infinity. So essentially, this curve comes with an infinite slope.

Now, it will turn out that this is the scenario that we have in three dimensions. If you are in two dimensions, then what you need to do is look at  $f$  that corresponds to  $m$  equals  $2/2$  or  $1$ , and in that case, the function diverges. So then, you have no problem in finding some intersection point for any combination of  $n \lambda^3$  over  $g$ . But currently in three dimensions, we have a problem because for  $n \lambda^3$  over  $g$  that falls higher than  $\zeta(3/2)$ , it doesn't hit the curve at any point, and we have to interpret what that means.

So for  $d$  equals  $3$ , encounter singularity when  $n \lambda^3$  over  $g$  is greater than or equal to  $\zeta(3/2)$ . This corresponds at a fixed density to temperatures that are less than some critical temperature that depends on  $n$ , and that I can read off as being  $1/kb$ . This is going to give me a combination.  $\lambda^3$  is proportional inversely to  $3/2$ , so this will give me  $n z^{3/2}$  over  $g$  to the  $2/3$ , and then I have  $h^2$  squared  $2\pi m$ . I put the  $kb$  over here.

So the more dense your system is, the lower-- I guess that's why I have it the opposite way. It should be the more dense it is, the higher the temperature. That's fine.

But what does that mean? The point is that if we go and look at the structure that we have developed, for any temperature that is high enough, or any density that is low enough, so that I hit the curve on its continuous part, it means that I can find the value of  $z$  that is strictly less than  $1$ , and for that value of  $z$ , I can occupy states according to that probability, and when I calculate the net mean occupation, I will get the actual density that I'm interested in.

As I go to temperatures that are lower, this combination goes up and up. The value of  $z$  that I have to get gets pushed more towards  $1$ . As it gets pushed more towards  $1$ , I see that  $\mu$  going to  $0$ , the state that corresponds to the lowest energy,  $k$  equals  $0$ , gets more and more occupied.

But still, nothing special about that occupation except what happens when I am at higher values. The most natural thing is that when I am at higher values, I should pick  $z$  equals 1 minus a little bit. And if I choose  $z$  to be 1 minus a very small quantity, then when I do the integration over here, I didn't get a value for the density, which is  $g$  over  $\lambda$  cubed, the limiting value of this  $f$  function, which is zeta of  $3/2$ .

I will call this  $n$  star because this is strictly less than the total density that I have. That was the problem. If I could make up the total density with  $z$  equals 1, I would be satisfied. I would be making it here, but I'm not making it up with the spectrum that I have written over here.

But on the other hand, if  $\epsilon$  is incredibly small, what I find is that the occupation number of the  $k$  equals 0 state-- let's write it  $n$  of  $k$  equals 0. What is that? It is 1 over  $z$  inverse, which is the inverse of this quantity, which is 1 plus  $\epsilon$ , which is approximately  $e$  to the  $\epsilon$  minus 1.

Actually, maybe what I should do is to write it in the form  $e$  to the  $\beta\mu$  minus 1 and realize that this  $\beta\mu$  is a quantity that is very small. It is the same thing that I was calling  $\epsilon$  before. And if it is very small, I can make it to be 1 over  $\beta\mu$ .

So by making  $\mu$  arbitrarily close to the origin or  $\epsilon\mu$  arbitrarily close to 1, I could in principle pump a lot of particles in the  $k$  equals 0 state, and that does not violate anything. Bosons, you can put as many particles as you like in the  $k$  equals 0 state. So essentially, what I can do is I can make this, but you say, well, isn't this already covered by this curve that you have over here? Are you doing something different?

Well, let's follow this line of thought a little bit more. How much do I have to put here? This  $n$  is the total number of particles divided by volume, and so this is going to be this quantity,  $n$  star, which is  $g$  over  $\lambda$  cubed divided by volume. Let's write it in this fashion,  $z$  over  $\lambda$  cubed zeta of  $3/2$  and then whatever is left over, and the leftover I can write as  $n_0$ .

So what we know is if we take  $z$  equals 1, this function will tell me how many things I have put in everything except  $k$  equals 0 that I will calculate separately. That amount is here. And I will put some more in  $k$  equals 0, and you can see that the amount I have to put there is going to be  $n - g$  over  $\lambda^3 z^{\frac{3}{2}}$ . Actually, I will have to put a volume here.

Why is that? The reason I have to put the volume is because the volume that I had here, that ultimately I divided the number of particles by volume to get the density, came from replacing the sum with an integration. So what I have envisioned now is that there are all of these points that correspond to different  $k$ 's. I'll replace the sum with an integration, but then there was one point,  $k$  equals 0, that I am treating separately. When I'm treating that separately, it means that is really is the same as the total number expectation value without having to divide by the volume, which comes from this density of state, and I have something like this.

Now, the problem is that you can see that suddenly, you have to pick a value of  $\mu$  that is inversely proportional to the volume of the system. This is a problem in which the thermodynamic limit is taken in this strange sense. You have all of these potential values of the energy,  $\epsilon$  of  $k$ , that correspond to  $\frac{\hbar^2 k^2}{2m}$ . There is one that is at 0, and then choosing  $k$  equals  $\frac{2\pi}{l}$ , you will have one other state, another state, all of these states.

All of these states are actually very finely spaced. The difference between the ground state and the first excited state is  $\frac{\hbar^2}{2m} \left(\frac{2\pi}{l}\right)^2$ . So this distance over here is of the order of  $\frac{1}{l^2}$ .

But I see that in order to occupy this with the appropriate number, I have to choose my chemical potential to be as close as  $\frac{1}{V}$  over the volume. So in the limit where  $l$  take the size of the system go to infinity, this approaches this. It never touches it, but the distance that I have here is much, much less than the distance that I have in the spacings that I made this replacement. So I can indeed treat these separately. I can give a particular weight to this state where the  $\mu$  has come this close to it as  $\frac{1}{V}$ , treat all of the other ones as part of this replacement of the summations with



the integral.

You can also see that this trick is going to be problematic if I were to perform it in two dimensions, because rather than  $1$  over  $l$  to the third power, that is, the volume in three dimensions, I would have had  $1$  over  $l$  squared. And indeed, that's another reason why two dimensions is special. And as I said, in two dimensions, the curve actually does go all the way to infinity. You won't have this problem.

So this is essentially what happens with this Bose-Einstein condensation, that the occupation number of the excited state for bosons is such that you encounter a singularity in three dimensions beyond a particular density or temperatures lower than a certain amount or highly degenerate case, you have to separately treat all the huge number of particles that is proportional to the volume that have now piled up in the single state,  $k$  equals  $0$ , the ground state in this case, but whatever the ground state may be for your appropriate system, and all the other particles go in the corresponding excited states.

Now, as you go towards zero temperature, you can see that this combination is proportional to temperature goes to zero. So at  $0$  temperature, essentially all of the particles will need to be placed in this  $k$  equals  $0$  state. So if you like, a rough picture is that the net density. This is the density,  $n$ , at high temperature is entirely made up as being part of the excited state. Let's draw it this way.

When you hit  $T_c$ , you find that the excited state can no longer accommodate the entire density. This is a coordination that, as we said, goes to zero at zero temperature as  $T$  to the  $3/2$ . So there's a curve that goes like  $T$  to the  $3/2$ , hits  $1$  at exactly  $T_c$ , which is a function of the density that you choose. This is the fraction that corresponds to the excited states. And on top of that, there is a macroscopic fraction that is occupying the ground state, which is the compliment to this curve. Basically, it behaves something like this.

One thing that I should emphasize is that what I see here makes it look like at high temperatures, there is no occupation of  $k$  equals  $0$  state. That is certainly not correct because if you look at this function at any finite  $z$  as a function of  $\epsilon$ ,

you can see that the largest value of this function is still at epsilon equals 0. It is much more likely that there is occupation of the ground state than any other state, except that the fraction that is occupying here, when I divide by the total number, goes to zero. It becomes macroscopic when I'm below the BEC transition.

Now, the properties of what is happening in this system below  $T_c$  is actually very simple because below  $T_c$ , the chemical potential in the macroscopic sense is stuck at 1. Essentially, what I'm saying is that in this system, the chemical potential comes down. It hits  $T_c$  of  $n$ , and then it is 0. Now of course, 0, as we have discussed here, if I put a magnifier here and multiply by a factor of  $n$  or a factor of volume, then I see that there is a distinction between that and 0. So it doesn't quite go to 0, but the distance is of the order of  $1$  over volume or  $1$  over the number of particles. Effectively, from the thermodynamic perspective, it is 0.

But again, for the purposes of thermodynamics, pressure, everything comes from the particles that are moving around. The particles in the ground state are frozen out. They don't do anything. So the pressure is very simple.  $\beta p$  is  $g$  over  $\lambda$  cubed,  $f_{5/2}$ , eta of  $z$  equals 1, which is some number. It is  $g$  over  $\lambda$  cubed, the zeta function at  $5/2$ , which again has some value that you can read of in tables, 1.34, something like that.

So you can see that this  $\lambda$  cubed is inversely proportional to  $t$  to the  $3/2$ , so pressure is proportional to  $t$  to the  $5/2$ . If I were to plot the pressure as a function of temperature, what I find is that at low temperatures, the pressure is simply given as  $t$  to the  $5/2$ . And you can see that this pressure knows nothing about the overall density of the system because this is a pure number. It is only a function of temperature, mass, et cetera. There is no factor of density here.

So what is happening? Because we certainly know that when I am at very high temperatures, ideal gas behavior dictates that the pressure should be proportional to temperature times density. So if I have two different gases where the density here is greater than the density here, these will be the form of the isotherms that I have at high temperatures.

So what happens is that you will start with the ideal gas behavior. We saw that for the case of bosons, there is some kind of an effective attraction that reduces the pressure, so the pressure starts to go down. Eventually, it will join this universal curve at the value that corresponds to  $T_c$  of  $n^2$ .

Whereas if I see what's happening with the lower density, at high temperatures, again, I will have the linear behavior of the ideal gas with a lower slope because I have a lower density. As I go to lower temperatures, quantum corrections will reduce the pressure. Eventually, I find that this curve will join my universal curve at the point that would correspond to  $T_c$  of  $n^1$ , which in this case would be lower. But beyond that, it will forget what the density was, the pressure would be the same for all of these curves.

And I have kind of indicated, or tried to indicate, that the curves join here in a manner that there is no discontinuity in slope. So an interesting exercise to do is to calculate derivatives of this pressure as a function of temperature and see whether they match coming from the two sides. And you will find that if you do the algebra correctly, there is a matching of the two.

Now, if I have the pressure as a function of temperature, then I also have the energy as a function of temperature, right? Because we know that the energy is  $\frac{3}{2} PV$ . So if I know  $P$ , I can immediately know what the energy is. I know, therefore, that in the condensate, the energy is proportional to  $T$  to the  $\frac{5}{2}$ . Take a derivative, I know that the heat capacity will be proportional to  $T$  to the  $\frac{3}{2}$ .

That  $T$  to the  $\frac{3}{2}$  is a signature of this condensate that we have. Again, heat capacity will go to 0 as  $T$  goes to 0, as we expect. And as opposed to the case of the fermions, where the vanishing of the heat capacity was always linear, the vanishing of the heat capacity for bosons will depend on dimensions, and this vanishing as  $T$  to the  $\frac{3}{2}$  is in general  $T$  to the  $\frac{d}{2}$ , as can be shown very easily by following this algebra.

I wanted to do a little bit of work on calculating the heat capacity because its shape is interesting. Let's write the formula a little bit more accurately. I have  $\frac{3}{2} v$ .

Pressure is  $kTg$  over  $\lambda^3$ ,  $f^{5/2}$  plus of  $z$ . This formula is valid, both at high temperatures and at low temperatures. At high temperatures,  $z$  will be varying as a function of temperature.

Let me also write the formula for the number of particles. The number of particles is  $V$  times  $g$  over  $\lambda^3$   $f^{3/2}$  plus of  $z$ . Once more, I can get rid of a number of things by dividing these two. So the energy per particle is  $3/2 kT$ ,  $f^{5/2}$  of  $z$  divided by  $f^{3/2}$  of  $z$ . Let's make sure I didn't make any mistake.

Now, I want to calculate the heat capacity, let's say, per particle, which is  $d$  by  $dT$  of the energy per particle. Now, I have one factor out here, which is easy to evaluate. There's an explicit temperature dependence here, so I will get  $3/2 kb$ . I will have  $f^{5/2}$  plus of  $z$  divided by  $f^{3/2}$  plus of  $z$ . Something here that I don't like.

Everything that I have written here is clearly valid only for  $T$  greater than  $T_c$  because for  $T$  less than  $T_c$ , I cannot write  $n$  in this fashion. So what I'm writing for you is correct for  $T$  greater than  $T_c$ . Point is that there is also an implicit dependence on temperature because  $z$  is a function of temperature.

So what I can do is I can do  $3/2 kbT$ , and this is a function of  $z$ , and  $z$  is a function of temperature. So what I have it is  $dz$  by  $dT$  done at constant volume or number of particles times the derivative of this function with respect to  $z$ . Now, when I take the derivative with respect to  $z$ , I know that I introduce a function with one lower derivative up to a factor of  $z$ . So I divide by the  $1$  over  $z$ , and then the derivative of the numerator is  $f^{3/2}$  plus divided by  $f^{3/2}$  plus in the denominator minus the derivative of the denominator, which is  $f^{1/2}$  plus of  $z$  times the numerator divided by  $f^{3/2}$  plus of  $z$  squared.

So what you would need to evaluate in order to get an expression that is meaningful and we can eliminate  $z$ 's as much as possible is what  $dz$  by  $dT$  is. Well, if this is our formula for the number of particles and the number of particles or density is fixed, then we take a derivative with respect to temperature, and I will get that  $dN$  by  $dT$ , which is  $0$ , is  $v$ . And then we have to take the temperature derivative of this combination,  $g$  over  $\lambda^3$ ,  $f^{3/2}$  plus of  $z$ .

Now,  $\lambda^3$  scales like  $1/T^{3/2}$ , so when I take a derivative of this, I will get  $3/2 T^{-5/2}$ , which I can again combine and write in this fashion,  $1/T^{3/2} + f^{-3/2}$  plus of  $z$ . Or then I take the implicit derivative that I have here, so just like there, I will get  $dz/dT$  at constant density times  $1/z$ . The derivative of  $f^{-3/2}$  will give me  $f^{-5/2}$  plus of  $z$ . Setting it to zero, we can immediately see that this combination,  $T/z \cdot dz/dT$  at constant density over the number of particles is minus  $3/2$ ,  $f^{-3/2} + z$  divided by  $f^{-5/2} + z$ .

So what I need to do is to substitute this over here. And then when I want to calculate the various limiting behaviors as I start with the high temperature and approach  $T_c$ , all I need to know is that  $z$  will go to 1.  $f^{-1/2}$  of 1 is divergent, so this factor will go to 0. This factor will disappear, and so then these things will take nice, simple forms.

Once you do that-- we will do this more correctly next time around-- we'll find that the heat capacity of the bosons in units of  $k_B$  as a function of temperature has a discontinuity at  $T_c$  of  $n$ . It approaches the classical result, which is  $3/2$  at high temperatures. At low temperatures, we said it is simply proportional to  $T^{3/2}$ , so I have this kind of behavior just a simple  $T^{3/2}$  curve.

And one can show through arguments, such as the one that I showed you above, that the heat capacity is continuous but its derivative is discontinuous at  $T_c$ . So the behavior overall of the heat capacity of this Bose gas is something like this. So next time, we'll elucidate this a little bit better and go on and talk about experimental realizations of Bose-Einstein condensation.