Fluctuations

1. *The Higgs mechanism:* Consider an n-component vector field $\vec{m}(\mathbf{x})$ coupled to a scalar field $A(\mathbf{x})$, through the effective Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + e^2\vec{m}^2A^2 + \frac{L}{2}(\nabla A)^2 \right],$$

with K, L, and u positive.

(a) Show that there is a saddle point solution of the form $\vec{m}(\mathbf{x}) = \overline{m}\hat{e}_\ell$ and $A(x) = 0$, and find $\overline{m}$ for $t > 0$ and $t < 0$.

(b) Sketch the heat capacity $C = \partial^2 \ln Z/\partial t^2$, and discuss its singularity as $t \to 0$ in the saddle point approximation.

(c) Include fluctuations by setting

$$\begin{cases} \vec{m}(\mathbf{x}) = \big(\overline{m} + \phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \phi_t(\mathbf{x})\hat{e}_t, \\ A(\mathbf{x}) = a(\mathbf{x}), \end{cases}$$

and expanding $\beta\mathcal{H}$ to quadratic order in ϕ and a.

(d) Find the correlation lengths ξ_ℓ, and ξ_t, for the longitudinal and transverse components of ϕ, for $t > 0$ and $t < 0$.

(e) Find the correlation length ξ_a for the fluctuations of the scalar field a, for $t > 0$ and $t < 0$. (The field A acquires a correlation length (mass) due to spontaneous symmetry breaking of the (Higgs) field $\vec{m}$.)

(f) Calculate the correlation function $\langle a(\mathbf{x})a(\mathbf{0})\rangle$ for $t > 0$.

(g) Compute the correction to the saddle point free energy $\ln Z$, from fluctuations. (You can leave the answer in the form of integrals involving ξ_ℓ, ξ_t, and ξ_a.)

(h) Find the fluctuation corrections to the heat capacity in (b), again leaving the answer in the form of integrals.

(i) Discuss the behavior of the integrals appearing above schematically, and state their dependence on the correlation length ξ, and cutoff Λ, in different dimensions.

(j) What is the critical dimension for the validity of saddle point results, and how is it modified by the coupling to the scalar field?

****

2. *Random magnetic fields:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + u\,m^4 - h(\mathbf{x})m(\mathbf{x})\right] \quad ,$$

where $m(\mathbf{x})$ and $h(\mathbf{x})$ are scalar fields, and $u > 0$. The random magnetic field $h(\mathbf{x})$ results from frozen (quenched) impurities that are independently distributed in space. For simplicity $h(\mathbf{x})$ is assumed to be an independent Gaussian variable at each point $\mathbf{x}$, such that

$$\overline{h(\mathbf{x})} = 0, \qquad \text{and} \qquad \overline{h(\mathbf{x})h(\mathbf{x}')} = \Delta\delta^d(\mathbf{x}-\mathbf{x}'), \tag{1}$$

where the over-line indicates (*quench*) averaging over all values of the random fields. The above equation implies that the Fourier transformed random field $h(\mathbf{q})$ satisfies

$$\overline{h(\mathbf{q})} = 0, \qquad \text{and} \qquad \overline{h(\mathbf{q})h(\mathbf{q}')} = \Delta(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}'). \tag{2}$$

(a) Calculate the quench averaged free energy, $\overline{f_{sp}} = \overline{\min\{\Psi(m)\}_m}$, *assuming a saddle point solution with uniform magnetization* $m(\mathbf{x}) = m$. (Note that with this assumption, the random field disappears as a result of averaging and has no effect at this stage.)

(b) Include fluctuations by setting $m(\mathbf{x}) = \overline{m} + \phi(\mathbf{x})$, and expanding $\beta\mathcal{H}$ to second order in ϕ.

(c) Express the energy cost of the above fluctuations in terms of the Fourier modes $\phi(\mathbf{q})$.

(d) Calculate the mean $\langle\phi(\mathbf{q})\rangle$, and the variance $\left\langle|\phi(\mathbf{q})|^2\right\rangle_c$, where $\langle\cdots\rangle$ denotes the usual thermal expectation value *for a fixed* $h(\mathbf{q})$.

(e) Use the above results, in conjunction with Eq.(2), to calculate the quench averaged scattering line shape $S(q) = \overline{\langle|\phi(\mathbf{q})|^2\rangle}$.

(f) Perform the Gaussian integrals over $\phi(\mathbf{q})$ to calculate the fluctuation corrections, $\delta f[h(\mathbf{q})]$, to the free energy.

$$\left(\text{Reminder}: \qquad \int_{-\infty}^{\infty} d\phi d\phi^* \exp\left(-\frac{K}{2}|\phi|^2 + h^*\phi + h\phi^*\right) = \frac{2\pi}{K}\exp\left(\frac{|h|^2}{2K}\right) \qquad \right)$$

(g) Use Eq.(2) to calculate the corrections due to the fluctuations in the previous part to the quench averaged free energy $\overline{f}$. (Leave the corrections in the form of two integrals.)

(h) Estimate the singular t dependence of the integrals obtained in the fluctuation corrections to the free energy.

(i) Find the upper critical dimension, d_u, for the validity of saddle point critical behavior.

****

3. *Long–range interactions:* Consider a continuous spin field $\vec{s}(\mathbf{x})$, subject to a long–range ferromagnetic interaction

$$\int d^d\mathbf{x} d^d\mathbf{y} \frac{\vec{s}(\mathbf{x}) \cdot \vec{s}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{d+\sigma}},$$

as well as short-range interactions.

(a) How is the quadratic term in the Landau-Ginzburg expansion modified by the presence of this long-range interaction? For what values of σ is the long-range interaction dominant?

(b) By estimating the magnitude of thermally excited Goldstone modes (or otherwise), obtain the lower critical dimension d_ℓ below which there is no long–range order.

(c) Find the upper critical dimension d_u, above which saddle point results provide a correct description of the phase transition.

****

8.334 Statistical Mechanics II: Statistical Physics of Fields
Spring 2014