Assignment #8

Due: Monday, April 22

1. **Classical Model of the Light Force**
   In class, we will discuss light forces using the OBE. In preparation for this, you are asked here to consider a (semi-)classical derivation. Assume that a hydrogenic atom can be modeled classically as an electron harmonically bound to a nucleus, with a resonant frequency ω_0 and damping coefficient γ. The nucleus is fixed at position $\mathbf{x}_0$ while the electron's position is denoted by $\mathbf{x}$. Now suppose the atom is illuminated with an electromagnetic wave of the form

$$\mathbf{E}(\mathbf{x}, t) = \hat{\epsilon} E_0(\mathbf{x}) \cos(\theta(\mathbf{x}) + \omega t) \tag{1}$$

   where $\theta(\mathbf{x})$ is the phase of the wave as a function of position $\mathbf{x}$ at time $t = 0$. The dipole moment of the electron may be written as

$$\mathbf{p}(\mathbf{x}, t) = \hat{p}(u \cos(\theta(\mathbf{x}) + \omega t) - v \sin(\theta(\mathbf{x}) + \omega t) \tag{2}$$

   Then the force of the light on the atom is

$$\mathbf{F} = (\mathbf{p} \cdot \hat{\epsilon}) \nabla E(\mathbf{x}, t) \tag{3}$$

   (a) **Time averaged force**
   Make the dipole approximation that $\mathbf{E}(\mathbf{x}) \approx \mathbf{E}(\mathbf{x}_0)$. Show that the time averaged force is

$$\langle \mathbf{F} \rangle = \frac{1}{2}(\hat{p} \cdot \hat{\epsilon})(u \nabla E_0(\mathbf{x}_0) + v E_0(\mathbf{x}_0) \nabla \theta(\mathbf{x}_0)) \tag{4}$$

   This expression is exactly analogous to the quantum-mechanically derived force. The first term is the dipole (stimulated) force, and the second term is the scattering (spontaneous) force.

   (b) **The potential picture**
   Recalculate the time averaged force on the atom from the instantaneous potential energy of a dipole in an electric field. How does this answer differ from that of 1a? Speculate as to why.

   (c) **Dipole moment of electron**
   Now we will solve explicitly for the dipole moment of the electron. In complex notation, the equation of motion is

$$m \frac{\partial^2 \mathbf{r}}{\partial t^2} + \gamma \frac{\partial \mathbf{r}}{\partial t} + m \omega_0^2 \mathbf{r} = -e \hat{\epsilon} E_0(\mathbf{x}_0) e^{i(\theta(\mathbf{x}_0) + \omega t)} \tag{5}$$

   where $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$. Solve this equation to find $\mathbf{p} = -e\mathbf{r}$. Substitute the quadrature components of $\mathbf{p}$ into the force equation from part (a) to find that

$$\mathbf{F} = -\frac{e^2}{2m\omega_0} \frac{\delta \nabla E_0^2 + \Gamma E_0^2 \nabla \theta}{4\delta^2 + \Gamma^2} \tag{6}$$

   where $\delta = \omega - \omega_0$ and $\Gamma = \gamma/m$. Make the approximation that $\omega \approx \omega_0$.

(d) **Force on a two-level atom**

The quantum mechanical expression for the light force on a two-level atom in the low intensity limit is

$$\mathbf{F} \approx -\frac{\hbar\delta\nabla\omega_R^2 + \hbar\Gamma\omega_R^2\nabla\theta}{4\delta^2 + \Gamma^2} \tag{7}$$

where ω_R is the Rabi frequency. Show that if we introduce the oscillator strength f_{fi} between the two levels, we may write

$$\mathbf{F}_{\text{quantum}} = f_{fi}\mathbf{F}_{\text{classical}} \tag{8}$$

2. **Master equation for a damped optical cavity**

A Fabry-Perot cavity can be modeled as being made of a high reflectivity mirror and a perfect mirror with fixed spacing. Clearly, photons stored inside this cavity will gradually leak out the partially reflecting mirror, causing the state inside to change. This process is described by a master equation, much like an atom coupled to fields is described by the optical Bloch equation. In this problem, we explore a simple derivation of such a master equation, for a single mode cavity.

Let a and $a^\dagger$ describe the optical mode of interest within the cavity, with characteristic energy $\hbar\omega$, described by the Hamiltonian $H_0 = \hbar\omega a^\dagger a$. Let $|\psi\rangle$ be the initial cavity state. Let us suppose that photons leak out of the cavity at a rate proportional to the photon number in the cavity and to Γ, which parameterizes the leakiness of the leaky mirror. Thus the probability that a photon leaks out within the infinitesimal timestep between t and $t+dt$ is given by $dp = \Gamma dt\langle\psi|a^\dagger a|\psi\rangle$. And with a probability $1-dp$ it doesn't. For these two cases we model the time evolution as follows:

- If a photon leaks out, the cavity state becomes $|\tilde{\psi_1}\rangle = a\sqrt{\Gamma\, dt}|\psi\rangle$.
- If no photon leaks out, the state becomes $|\tilde{\psi_0}\rangle = e^{-iHt/\hbar}|\psi\rangle$ having evolved under the "Hamiltonian" $H = H_0 - i(\hbar\Gamma/2)a^\dagger a$. Note that H is not Hermitian. It has an imaginary term, so it is "lossy". Evolving a state forward in time with H yields an un-normalized state.

The two un-normalized states $|\tilde{\psi_0}\rangle$ and $|\tilde{\psi_1}\rangle$ will later be two "components" of the density matrix.

a) Compute the normalized states $|\psi_0\rangle$ and $|\psi_1\rangle$. Provide some words of justification for the lossy term in H (for instance, why is it proportional to $a^\dagger a$?).
Where you encounter exponentials of H or $H^\dagger$ (which is NOT equal to H), expand them to first order in dt.

b) At time t, suppose the cavity starts in the pure quantum state $\rho(t) = |\psi\rangle\langle\psi|$. At time $t+dt$ the state of the leaky cavity system is given by

$$\rho(t+dt) = (1-dp)|\psi_0\rangle\langle\psi_0| + dp|\psi_1\rangle\langle\psi_1|\,. \tag{9}$$

Rewrite this density matrix, building it out of the the un-normalized states described in part a). The only bras or kets to appear in your final expression should be $|\psi\rangle$ and $\langle\psi|$.

c) Compute $\rho(t+dt) - \rho(t)$ for small dt, and write the coarse-grained differential equation

$$\frac{d}{dt}\rho(t) \approx \frac{\rho(t+dt) - \rho(t)}{dt}\,. \tag{10}$$

Show that this has the proper Lindblad form of

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0,\rho] - \frac{1}{2}\left(C^\dagger C\rho - 2C\rho C^\dagger + \rho C^\dagger C\right)\,, \tag{11}$$

and identify what C and $C^\dagger$ are for this case.

MIT OpenCourseWare
http://ocw.mit.edu

8.422 Atomic and Optical Physics II
Spring 2013

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.