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**PROFESSOR:** Good afternoon. The title here is the topic for our class today. We want to discuss formally the derivation Optical Bloch Equations, but usually when I teach you something, I have a general concept in mind and the concept right now here is, how can we get from unitary time evolution of a quantum system to rate equations anticipation. And this is sort of the subject of master equation open system dynamics, and this cartoon sort of tells you what you want to do.

We have a total system, which is one part we are interested in, and the other one, often it has many, many degrees of freedom, and we don't want to keep track of them. But they have one Hamiltonian. But we are only interested in how our atomic system evolves, and we want to find an equation, which is no longer a Schrodinger equation. How does an initial density matrix describing our system develop with time? And as we will see, in general, it follows a master equation, and we want to discuss what are general principles of such equations.

But it's very important here is that we are not keeping track of what happens in the environment, and that's associated here with a little bucket or trash can. Every result photons or such are measured, the environment is constantly projected on a measurement basis, and therefore, re-introduce probabilistic element into the part we don't observe mainly the part which characterizes the atoms.

So for that we need the formalism of density matrix, and that's where we want to start. At the end of last lecture, I reminded you that the density matrix can always be written as an ensemble where you say you have a certain probability for certain rate function. This is always possible, but it is not unique. So each of you could actually create the same density matrix by preparing a number of quantum states with a certain probability and saying this is my density matrix.

And also each of you has prepared different quantum states. If you sum them up in that same way, you get the same density matrix and therefore, all observables or measurements you will do on your ensemble in the future will be identical because the density matrix is a full description of the system. So at that level, it may look sort of very trivial that we have different unraveling. Unraveling means you look microscopically what is behind the density matrix, but in your homework assignment, you will all show that you can have a system which have very, very different dissipation mechanisms, but they're described by the same equation.

So therefore, it's physically not possible to distinguish by just measuring the density matrix what causes a dissipation. Of course, if you know what causes a dissipation, fluctuating fields or collisions, you know more than the density matrix knows. Any questions about density matrix or the agenda we want to go through today?

Just a quick reminder and since this was covered in 8421, and most of you know about it, I pre-wrote the slides. The density matrix is a time evolution. There is one which is sort trivial and covered in more elementary takes, and this is the Hamiltonian evolution, and I'm sure you've seen it. The time evolution, the unitary time evolution involves the commutative of the Hamiltonian with a density matrix, and later on, we want to specialize including dissipation, including the environment to the evolution of a tool evolutionary system driven by a monochromatic field

And this the famous Jaynes-Cummings Hamiltonian, and we can characterize this system by a density matrix, but will be very important is that we distinguish between populations, the diagonal parts, and the coherences. And if you simply put this density matrix into this equation, you find the time evolution of the system, and we will refer to that result later on. Many of you have seen it in 8421. we can parametrize the density matrix for the two-level system.

So we can parametrize the densities of the two-level system by a local vector,  $r$ , which is defined by this equation. And then the equation of motion is simply the rotation of a Bloch vector on the Bloch sphere, and it has a rotational axis, which is given by-- the undriven system rotates around the z-axis. This is just  $e$  to the  $i$

omega naught t, the normal evolution of the free system. But if you divide it with a monochromatic field rotation over the x-axis and the x-axis, of course, can take your two-level system, and flip it from the ground to the excited state. So it's just a reminder of the simple unitary time evolution, but now we want to add dissipation on top of it.

And what I've decided that before I discuss with you Optical Bloch Equation and master equation in general, I want to give you a very simple model. I really like simple models which capture the essence of what you are going to discuss. So what I want to use is I want to use a beam splitter model I formulated for photons, but it would also immediately apply to atoms.

And this model, what I like about it, it has all the ingredients of integration of the master equation we'll do later on without the kind of many indices and summations and integration, but it captures every single bit of what is important. And I usually like to present exactly solvable, simple models where you get it, and then I can go a little bit faster for the general derivation because you know exactly what the more complicated equations, what they are doing.

So in other words, what I want to derive for you is we want to have the following situation. We have a beam splitter, and we know everything about beam splitters because we talked about them in the first part of the course, and we have a wave function, which is the input, and this is a photon. And we want to understand after the beam splitter, how has the system evolved. In general, it will be a density matrix, and what you want to find out is, what is the equation for the density matrix.

Maybe this density matrix goes through the next beam splitter and then we want to know what comes out of it. And all we have to apply is the formalism we developed for the beam splitter earlier in the course. Of course, the beam splitter is not as harmless as it looks like. There is another part and another part one here brings in the environment, and for the environment, we will use the vacuum. That's the simplest environment.

It's actually important environment because it is the environment we will use all the

time when we discuss spontaneous emission. We send photons into the nirvana, into the vacuum and they disappear, and this is our modified. But the other one which is often not so explicit. If you send your photons away, you're not keeping track of what happens, but you could as well perform a measurement, and this is what we put in here.

We say those photons hit a bucket or detector and measure them, we observe them. There's nothing else we do with them, so we can as well measure them, and this being immediately lead to the equation for the density matrix. So this is what you want to discuss, and it will have all the ingredients later on in a mathematically simple form for the derivation of master equation.

So let's consider a similar photon for that. So the wave function, this is superposition of no photon and 1 photon, and the coefficients are alpha and beta. And just for simplifying notation, I pick alpha and beta to be real.

So what do we expect to happen at the beam splitter? Well, there's a probability that the photon gets reflected. Probability to reflect the photon and therefore to observe the photon. This probability is, of course, beta square-- the probability that we have a photon to begin with-- and then the beam splitter, remember we categorized the beam splitter with angle sine theta cosine theta. Sine theta was the reflection amplitude, cosine theta, the transmission amplitude.

So this probability, which I call  $P_1$  is the probability for reflection and for measurement. And now naively you would think what happens after the system has passed through the beam splitter, with a probability of  $P_1$ , we've measured the photon. We know for sure there is no photon left. The system is in the vacuum state, but then you would say, well, maybe with probability  $1 - P_1$ , we have not measured anything. Nothing has happened to the wave function, and that means the wave function just continuous.

Well as we will see, this is wrong. We are missing something. What we are actually missing is that if you measure nothing, the wave function is not sine. The possibility that we could have measured something, changes the wave function. I will comment

on that in much more detail in a few lectures down the road when I derive for you quantum Monte Carlo wave function. I will have a wonderful discussions with you about how does non-observation change a wave function.

So we will talk about the physics behind it in some more detail. Right now, I don't want to get into this discussion. I simply want to use our beam splitter equation, so we can just take the beam splitter equation and apply it.

So our output state is obtained by taking the operator for our beam splitter, and maybe you remember that the propagation for beam splitter was discovered by an operator, which had a dagger b dagger a in the exponent. a and b are the two input nodes. And the angle of the beam splitter, which interpolates between 0% and 100% reflection transmission is theta.

And we're now looking for the output state of the total system. We're not performing the measurement yet, and this is now acting on the total system, which is the cross product of our photon system, of our system of interest. And the other input, which we call the environment or the vacuum, is 0.

Well, look a few weeks back, we have done that all. The output state is, well, there was a probability, alpha, that we had no photon in the state psi. And if we have no photon in the state psi and no photon in the vacuum, this is the state 0, 0. What I denote here with this second place is the environment. And now we have one photon. We have exactly one photon with the amplitude beta, and this photon is split with cosine theta transmitted and with phi theta reflected.

If you transmit it, we have 1, 0. If we reflect it, we have 0, 1. And again, this is the environment, and here is a photon in the environment. So let me just be clear that this is where the environment comes in. It is a vacuum state, and here, this is the output part for the environment. This is where we do the measurement.

And I don't think it matters. I haven't really told you which is mode A, which is mode B. It doesn't matter, but one, let's say the environment is mode B, and the system evolves in mode A. As you can see, I'm using a new program, which has some nicer

features in terms of handwriting, but it is a little bit rough in scrolling, so I sometimes have to scroll back and forth.

So what is our output? Now, we have two possibilities. The environment is 0, or the environment is 1, and we perform a measurement. So we have to now go into a probabilistic description. So with probability  $P_1$ , we have done a measurement, and our output state is now the vacuum state.

With probability  $1 - P_1$ , we have not detected anything in the vacuum, and therefore, our state of the system is  $\alpha |0\rangle + \beta \cos \theta |1\rangle$ . Is  $\alpha |0\rangle$ , so it is not  $\beta |1\rangle$  as naively would have assumed. It's not the original state. There is a  $\cos \theta$  factor, which we got exactly from the beam splitter from the unitary evolution provided by the beam splitter. And since this state is no longer normalized, I have to normalize it by  $\alpha^2 + \beta^2 \cos^2 \theta$ .

So now we have done our measurement probability  $P_1$  to detect the photon. This projects the system into the vacuum state with the probability  $1 - P_1$ . We have that state. Just one second. Scroll in the pictures. Write that down. In fact, millions that our system is now described by a density matrix with probability  $P_1$  and  $1 - P_1$ .

With probability  $P_1$ , we are in the vacuum state, and with probability  $1 - P_1$ , we are in that state, the denormalized state  $\psi_{\text{naught}}$ , which I just hold down.

Question?

**AUDIENCE:** Have you considered  $\theta$  to be some like a dynamical phase evolution system. It's very low order like when you expand it the first time it looks almost identical to [INAUDIBLE] quantum effect maybe. The environment is measuring the state in some way, and I mean, it's the lowest order now.

**PROFESSOR:** Yeah, I quite agree that random 0 is just an example of it. Pretty much, it's all the same. Yeah. What we do here is, I like the beam splitter because the beam splitter provides an exact formulation of the measurement process. You really can use a

beam splitter to discuss what happens fundamentally when you perform a measurement. And the beam splitter is one typical implementation of that, but it has all the features you'll find in any measurement system.

And especially what you observe here, let me just emphasize is, the fact that when you do not make a measurement is changing the wave function from the initial wave function  $\psi$  to  $\psi$  naught, we have a factor of  $\cos \theta$  here. And that's also very general. A measurement perturbs, modifies your wave function no matter what the outcome of the measurement is.

So let me write it down because we want to take it to the next level. So we have now found in terms of the beam splitter, angle  $\theta$ , and the parameters of the initial state,  $\alpha$   $\beta$ . We found the density matrix after the beam splitter. Yes.

So what is the next step? Our goal is to derive the master equation for the density matrix, the time evolution of the density matrix. So since we want to discuss the time evolution, we want to find a differential equation. So what we want to figure out is, what is the difference between the output density matrix and the input density matrix. The input was, of course, pure state characterized by the matrix population  $\alpha^2$  and  $\beta^2$  of diagonal matrix element of  $\alpha$   $\beta$ .

The difference between the density matrix is can just calculate the difference. You can simplify things by applying some trigonometric identities, so this is an exact result,  $\cos \theta - 1$ . Here we have  $\alpha \beta \cos \theta - 1$ , and on the diagonal, we have  $\cos^2 \theta - 1$  divided by 2.

Anyway this is an intermediate result. We're interested in the differential equation. We want to sort of find out what happens when we observe, when we have the density matrix interacting with the environment all the time. And this can be simulated by beam splitters by using many beam splitters with a small degree of reflection.

So we want to simplify this result now for the case of many beam splitters, and each of them has a small tipping angle,  $\theta$ , and for later convenience, I defined  $\theta$

to be  $\gamma \text{ times } \Delta t \text{ over } 2$ . That's just my definition of  $\theta$ . So what we have in mind now is that we start with the system  $\psi$ , and we have many such beam splitters with an infinitesimal tipping angle. Each beam splitter has the vacuum at its input state. And we always perform the measurement.

If I take the equation above, which I know you can't see anymore, we find a differential equation for the density matrix, which is we find an infinitesimal change  $\Delta$  over the density matrix, which looks like this. So all I've done is, I've used the equation above, and I've done a Taylor expansion in the small angle  $\theta$ . And the reason why I brought in the square root, well, we get  $\cos \theta$ .

The first order Taylor expansion or the lowest order Taylor expansion from  $\cos$  is  $1 - \theta^2$ . So I get the square root squared. So I get  $\gamma$ , which appears here, and then I divide by  $\Delta t$ .

So this is just an exact mathematical expression, and the next step is to form a differential equation. But before I do that, I want to emphasize the two features we are using here. They sort of enter automatically, but these are the two big assumptions we make when we derive a master equation.

The first one is that we always have a vacuum state as the input. So in other words, the environment is always in the same state, which is a vacuum state, and this is sort of called a Bohr approximation. What it means is that we do a measurement here, but the vacuum is not changing. In other words, we are not overloading the vacuum with so many photons that suddenly the vacuum is no longer in the vacuum state.

Or in the case of spontaneous emission, the vacuum can just take as many photons as you dumping into it. They disappear so quickly that for all practical purposes, the environment stays in the vacuum state. So this is called the Bohr approximation. The environment is not changing. It has enough capacity you to be modified by the measurement process.

And the second thing which is related is, the vacuum is always in the same state,



and there are no correlations from here to here to here, there is no memory effect. Everything is completely uncorrelated. So the environment is uncorrelated. It has no memory. Its correlation function is a delta function, and this is called Markov approximation. So these are the two effects which are important. One is no memory for the environment, Delta function correlation, Markov approximation, and the two, of course, are related in the environment is all of this in the same state.

So remember this is a change for the density matrix, and alpha beta where the original parameters of the density matrix for the input state. So I can now rewrite everything as a differential equation. The density matrix has a derivative for the diagonal matrix elements and for the coherences. Here we have plus gamma. Here we have minus gamma, 0, 1, 1, makes sense because we conserve the trace. We have unity probability that we have a stellar system. And therefore the two diagonal matrix elements, the population, the sum of them cannot change with time.

And for the coherence, we have gamma over 2. And if you're familiar with Optical Bloch Equations, which we derive next, we can say that these means if 0 is the ground state that the ground state changes because-- call it spontaneous emission from the excited state. This equation would say that the excited state decays with the rate gamma, and sometimes you may have wondered about that there are factors of two appearing, which also appears here that when the excited state decays with a rate gamma, we have a factor here for the coherences, which is gamma over 2.

So what we have accomplished in contrast to let's say Einstein's equation with the Einstein a and b coefficient, which lead to rate equations for the population, we have now a new feature. We have an the equation for the coherences, and we find a decay of the coherences with half the rate as a decay of the population. Questions?

So if you want you could rewrite this model for photon, which goes through beam splitters, undergoes measurement. You can rewrite it from atomic wave function and you measure whether the atomics in the excited and ground state and the equation for the measurement performed on the atom is exactly as the equation by

which the beam splitter acts on the photon state. So what I've shown here it's very specific for a single photon because I could use simple equations, but everything is what you find in a much more general situation.

So before I give you the general derivation of the master equation, let me talk about what we have learned from this example and what the general procedure is. The first thing is our goal is to find a differential equation for the density matrix of the system.

I just remember. There was one thing I wanted to mention. In the previous derivation with the beam splitter, I started with a purer state, and the purer state developed into a statistical mixture, and this statistical mixture would then transform the next beam splitter into another statistical mixture. I derived the differential equation for you for the first state from the pure state to the statistical mixture. But if you would spend a few minutes, you could immediately show that you can start with an elementary density matrix, look how it evolves through the beam splitter, and you get exactly the same differential equation.

So the general procedure is, we want a differential equation how the density matrix evolves with time. And this will be obtained by finding an operator which acts on the initial density matrix. This operator is not a unitary operator because we are performing measurements through the environment, even if you don't actually perform them. Once we dump something into the environment, it's out of our control and anybody could go and perform a measurement, and so we should assume that this measurement has been taken. It's one of those quantum mechanical things that you don't even have to care whether somebody does it. The environment does it for you.

So this operator is called a Liouvillian operator. It's sometimes called-- and I haven't really traced down why-- it's called sometimes super operator. I know what superconductivity is, but I don't know what the super powers of this operator are, but that's just a name which you will find.

The second thing which we have used is that the evolution of this system can be

obtained from the time evolution of the total system by performing a trace over the degrees of freedom of the environment. This was exactly what we actually did when we said the system continues with probability  $P_0$  in one state and probability  $P_1$  in the other state. The operation which lead to this density matrix was exactly the partial trace.

So this is the second general feature which we have to implement. Thirdly, if we could do one and two exactly, we would have an exact formulation for a small part of a quantum system no matter how complicated the environment is. In practice, we can solve the equations only when we make simplifying assumptions about the environment. One is, it is large, and more important, therefore, it's unchanging, and this is the Bohr approximation.

And the second feature is, it has a short correlation time,  $\tau_c$ . In the beam splitter, I have made the assumption that there is no correlation between different beam splitters in the derivation, which I want to walk you through. Right now, you will see explicitly where the correlation times enter. And this is called the Markov approximation.

And finally, this is number four, the whole possibility to derive a master equation hinges on the fact that we have different time scales which are very different. We are interested in the evolution of our system. We want to know how it relaxes, and this is on a time scale  $1/\gamma$ .

So we call this slow. We are interested in the variation of our system, the atomic system or the photon state, which passes through the beam splitter, and this time scale has to be much slower than the fluctuations of the environment. So therefore, if the environment has fluctuations, which in the beam splitter model where assumed to be 0, it was delta function of time, if that correlation time is much smaller than the time it takes for the system to relax and to evolve, that opens a window  $\Delta t$ , and this is the time scale of the master equation.

Just to give you one example for the spontaneous emission, the correlation time,  $\tau_c$ , would be the time it takes the photon to disappear from the atom. And the

photon has disappeared from the atom when it is one wavelength away. So typically, the correlation time of the vacuum for spontaneous emission is one cycle of the optical frequency. It's very, very fast. Whereas typical decay times of the excited states, a nano second. It's six orders of magnitude slower, and this is what we describe.

But on a time scale of a femtosecond of one optical cycle, the photon has not detached from the atom and it could go actually back to the atom. During that time, we talked a little bit about it when we did this diagrammatic discussions of resonance scattering for very, very early times. You don't have exponential decay because you cannot do the approximations where we approximated the kernel by something which was completely energy dependent. And so what happens at such short times, we encounter here again in such short times, we will not have a simple description of the system.

So the last point, let me summarize. Our goal is that we describe the density matrix of the system, and we want to find the Liouville operator or some matrix which acts on it. And because we will integrate over time steps, which are larger than the correlation time of the system, we can also call it, it will be a coarse-grained evolution. Any questions about that?

I really like the discussion, the derivation of the master equation, how it is presented in atom-photon interaction. But it is presented on more than 50 pages with many, many equations. So after giving you all of the principles, all of the concepts, I want to go with you now over those equations and point out how the principles, which we encountered with the beam splitter, how they are now implemented in a very general context.

I will not be able to give you all of the mathematical aspects of it, but I think by now you know that the book atom-photon interactions are actually wonderful. You can get a lot of conception information out of it by looking at the equations without understanding every technical detail. So I would really encourage you, if there is something which piques your interest and I hope there will be things which you'll find

very interesting, that you go to the book and read it. So I'm exactly following that actually I used copies of the book.

So we have a Hamiltonian, which is describing the atomic system. It describes the reservoir and then there is an interaction. We keep it very general here, but you may always think well, the atom is your favorite two-level system. The environment is maybe the vacuum with all its possible modes, and the interaction is the dipole interaction or the  $\mathbf{a} \cdot \mathbf{p}$  interaction.

So we start out with an equation, which is nothing else than Schrodinger's equation for the density matrix. The time derivation of the density matrix is commutative with the Hamilton. But it is often useful and you've seen it many times, to go to the interaction representation, that the time dependence due to the unperturbed part of the operator is absorbed in a unitary transformation, so therefore, this density matrix in the interaction representation evolves not with,  $\hbar$ , because  $\hbar$  naught is taken care of, it only involves due to the coupling between the two systems, between the system and the environment.

So now this equation, we are interested in a time step  $\Delta t$ . And this time step,  $\Delta t$  remember, we want to coarse grain, will be larger than the correlation time of the reservoir, and you will see exactly where it comes about. So we want to now do one of those coarse-grain steps. We take this equation and we integrate from time  $t$  to time  $t + \Delta t$ .

So this is exact here. But now we want to iterate, and that means the following. We have expressed the time step in the density matrix by having the density matrix there. But now we can do in a first-order perturbation theory, we can do one step and we get the second order result by plugging the first order result into this equation.

It's the same we have seen with our diagrams and such. We have an exact equation. It's useless unless we do something, and what we do is, we realize that we can iterate it because the part we don't know involves one more occurrence of the interaction potential. And when you plug the  $n$ th order solution in here, you get

the  $n$  plus first order solution. And this is exactly what is done here.

And I skipped a few equations here. This is what he's done here, number one. And number two is, we are interested in the system, not in the reservoir, so therefore we perform the trace over the reservoir. And the trace of the reservoir for the photon beam splitter, we say we have two possible states, we detect a photon or not. And for the system, we have now a density matrix which is probability  $P_0$  in one state, probability  $P_1$  in the other state. And this is exactly what the operator partial trace does.

Remember also I want to really make sure that you recognize all the structures. The time evolution of a density matrix was a commutator with  $h$ , but in the interaction picture, it's a commutator with  $v$ . But since we are putting the first order result in here, the second order result is now the commutator of  $v$  with the commutator of  $v$  and  $o$ . It's just we have iterated one more time.

So the  $\tilde{\rho}$ , the density matrix for our system,  $\tilde{\rho}$  means in the interaction picture is now the partial trace over the reservoir of the total density matrix.  $\tilde{\rho}$  means in the interaction picture. And the important part here is that it is exact. We have not done any approximation here. Any questions?

Of course, now we have to make approximations because we cannot solve an interactive problem exactly. The first one is-- and what do we want to do in the end? We want to keep the first non-trivial term but to the extent possible, we want to factorize everything. We want to get rid of the entanglement of the environment in the system and only get sort of the minimum which is provided by the coupling.

So this evolves as follows. The interaction we assume is a product of two operators. One operator acts on the system, one operator acts on the environment. So this could be the dipole acting on the atom, the vacuum field,  $e$ , interacting with the environment or it could be  $p \cdot a$ . Or maybe your system has a magnetic moment,  $m$ , and the environment consists of fluctuating magnetic fields.

So we'll pretty much find in every kind of measurement that a measurement

involves the product of two operators. One is an operator for your system and one is an operator for the reservoir of the environment. And so this is one thing we want to use, and now there is one thing which the moment we will set in our equations, there is one thing which will naturally appear. Let me scroll back.

What we have here is the interaction operator,  $v$ , at two different times. So this means if something happens at different times and we integrate over times. This is a correlation function, a correlation function between  $v$  at the time  $t$  prime and the time  $t$  double prime and since the reservoir part of this interaction is the operator,  $r$ , so what we have here now is, we have a correlation between the operator,  $r$ , at two times, which characterizes the environment.

And now comes an important approximation. You remember I said we want to assume that the environment has a very short correlation time. Whenever a photon is emitted, it appears dramatically fast. It disappears in one optical cycle, and the environment is sort of reset, it's back in the vacuum state. So this is now expressed here that this product over which we take the partial trace has a very short coherence time.

And the fact is now the following. We are integrating over a coarse-grained step  $\Delta t$ , but this correlation function goes to 0 in a very short time. So therefore it will not contribute a lot. Let me write that down. What we are going to approximate is our total density matrix is now approximately factorizing in a density matrix describing the atomic system. Well, we describe the atomic system when we trace out the environment.

We describe the environment when we trace out the atomic part. And if we now form the direct product, we are back to the total system, but we have factorized the total density matrix into two parts. What we neglect here is a part which cannot be factorized which is the correlated part of it. But what happens is, since we are integrating over time steps  $\Delta t$  and the correlation decay in a very, very short time, the result is that this complicated part, which we could never calculate, is smaller than the first part by the ratio of the time where the correlations contribute

over the time  $\Delta t$ , the time step we are going to take.

So this is a very critical assumption. There is a whole page or two in the book where an photon-atom interaction they discuss the validity of this assumption, but I've given you the physical motivation that we indicate over much larger time, and if this time is large and the correlation is lost for short time, they only contribute with this small parameter to the result.

So in other words, this means after-- we have an interaction between the environment and the system. We write it down in second order, but the second order result is now we evaluated by factorizing the density matrix into our system in the reservoir. So that means in that sense if you factorize something, it looks as if it's not interacting, but the trick is the same.

You write down something to first and second order, and once you have factored out the important physics, now you can evaluate the expression by using an approximation, which is now the approximation that the density matrix factorizes. So with that, this is the approximation that we have made that the correlation time is very short.

And now we have a differential equation for the density matrix  $\sigma$ , which describes or atomic system. We have traced out the degrees of the reservoir. And now we want to insert B.17. You probably don't remember what B.17 is. It says that the interaction operator is a product of  $a$ , the operator  $a$  for the atoms and  $r$  for the reservoir.

So the reservoir part,  $\tau \tau'$ , gives a correlation function. This is the correlation function between the operator,  $r$ , at two different times and the part which acts on our system, the  $a$  part, is explicitly kept here.

So this is now a general master equation. It tells us the time evolution of the density matrix in this form. It looks very complicated, but this is because it's very general. In order to bring it into an easier form, we want to now introduce a basis of states, energy eigenstates of the unperturbed system, and write down all of these operator



into such a basis of states.

But anyway you saw here how we had an exact equation, and the main approximation we made is that the operator acting on the environment has a very short correlation time. Any questions?

Well, you're only a few minutes away from producing this result to Fermi's golden rule, which you have known for a long, long time. It's just we have made very general assumptions. You see sort of how the assumptions propagate, but now if you write it down for an energy eigenbasis, you will immediately see results you have probably known since your childhood.

So we want to have energy eigenstates of the atomic operators, so this is sort of ground and excited state if you think about a two-level system. The previous equation, I have to go back to it. Our previous equation is a differential equation for the density matrix here, and here is the density matrix.

So now we formulate this equation into an energy eigenbasis, and what do we get? Well, we get an equation for the matrix elements, and what matrix elements are important? diagonal matrix elements which are population of diagonal matrix elements which are coherences. So we pretty much take this equation, use the energy eigenbasis and look, what do we get for the populations and what do we get for the coherences.

So the structure is now the following. That we have our matrix elements  $\rho_{ab}$ . There is one part which looks like a unitary time evolution. This is what comes from the Hamilton operator. This is sort of the-- we'll see that in a moment-- but this is the time evolution without relaxation and now we have something here which are generalized relaxation coefficients. And you will find if you go further above that those relaxation coefficients are directly related to the correlation function of the reservoir.

So we can now specify what happens between populations. Population means that we have a differential equation, let's say between  $\rho_{aa}$  the puller, and  $\rho_{cc}$ ,

so we have a rate coefficient which connects the population in state A with the population in state C.

And if you take this expression, you find several things. Well, you find Fermi's golden rule, in a generalized way-- that's always nice-- you find Fermi's golden rule. When you integrate over time, you often get delta function, and you expect to get a delta function because of energy conservation. So you get that, of course, naturally.

Secondly, we have second order matrix element, which you know from Fermi's golden rule, but now we have the following situation that the matrix element in Fermi's golden rule may actually depend on the state,  $\mu$ , of the environment. So you have maybe 10 different possibilities for the environment, and Fermi's golden rule gives you spontaneous emission, which is different for those 10 states. And naturally, since we have performed the partial trace over the environment, we have all those rates weighted with the probability that the environment is in one of those states.

So what you find here is a simple generalization of Fermi's golden rule. And if you look at the off-diagonal matrix elements, for instance, you want to know what is this rate, what is the rate coefficient, which gives you the time derivative of the coherence, and it's multiplied with the coherence. You'll find now that in general, this rate coefficient has a damping term, but it may also have an imaginary term.

And I hope you remember when we played with diagrams, that we had something similar. There was something which we called the radiative shift. I called it the AC Stark effect of a single photon, and here it is a level shift which comes because the environment interacts with your system and it shifts the levels a little bit.

So in addition to just relaxation, spontaneous emission, and damping, there is also a dispersive part, a level shift, and it has exactly the same structure. Let me add that  $\Delta_{ab}$  is the difference between the shift of state b and the shift of state a. And those shifts have exactly the same structure. You have to take the principle part of something which has 1 over the difference of energies, and we discussed that this has to be understood by somewhere adding an infinitesimal imaginary part and

doing the right thing with complex function. It's actually related to Laplace time difference between Laplace transformation and Fourier transformation.

So anyway what I find sort of beautiful is that we started with a most general situation. We perform the partial trace. We made one assumption of short correlation times, and a lot of things we have known about quantum system just pops out in a very general form here. Any questions about so far?

Well, the coherences are, of course, more interesting than the population. Coherence is always something physicists get excited about it because it captures something which goes often beyond classical system that we have quantum mechanical coherences. And what happens is the coefficient here, which provides the damping of the coherence, just comes out of the formalism, has two parts. And one part is an adiabatic part and the other one is a non-adiabatic part.

Well, and that makes sense. If you have two quantum states and there is a coherence, some phase between the two, the phase can get lost if you do a transition between the two states or one state undergoes a collision and is quenched. So you definitely have one part which is due to the fact that the quantum states or the population changes, and you find that there is this state-changing part, which is pretty much the sum of all the rate coefficient leading out of state, leading to the decay of state a and leading to the decay of state b.

In other words, if you have a two-level system, which has a coherence and you have decay of the excited state and decay of the ground state, you would expect that those decay terms appear also in the decay of the coherence between the two levels and they do, and they appear with the correct factor of  $1/2$ .

But there is another possibility and this is the following. You can have no [INAUDIBLE] of the population of the state, but you can still lose the coherence. The model you should maybe make is that you have spin up, spin down. You are not perturbing the populations in spin up and spin down, but the environment provides fluctuating magnetic field. Then due to the fluctuating magnetic field, you no longer can keep track of the phase, and that means in your identity matrix the off-diagonal

matrix elements decay.

And we find that here this is the second part which in this book is called the adiabatic part, and the physics behind it is now pure de-phasing. So it's an independent way for coherences to decay independent of the decay of the population. Questions? Collin.

**AUDIENCE:** Where does the Markov approximation come in?

**PROFESSOR:** The Markov approximation is, so to speak, the delta function approximation, which would say that-- I mean, I introduced the correlation function between the reservoir operator and said the correlation time  $\tau_c$  is very, very short. The Markov approximation would actually state that it would actually say in a more radical way the correlation time is 0.

And the Bohr approximation, the fact that the reservoir is unchanged came in when we said the total density matrix for the second order expression just factorizes. It factorizes into the environment, which is just this density matrix of the environment. It's not changed by the interaction with the system. And this is the Bohr approximation. We just use the same expression for the reservoir independent of the measurements the reservoir has done.

Other questions? Yes, Nancy.

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** This is something very general. Thank you actually for the question. Whenever we have some damping of the population, the coherence is only damped with a factor of  $1/2$ . One way to explain it in a very simple way is, that if you have an amplitude  $\alpha$  excited and  $\alpha$  ground, the population in the excited state is this squared. So you sometimes make the model that the amplitude decays with  $\gamma/2$ , but the population is-- because you take the product, decays with  $\gamma$ .

So you would say  $\alpha_e$  and  $\alpha_g$  both decay with half the rate, but the probability is for the total rate, and the coherence is the product of the amplitudes.

So therefore when you look at the coherence, this decays with  $1/2 \gamma_e$ . This decays with  $1/2 \gamma_g$ , and this is what you get here,  $1/2 \gamma$  in state a,  $1/2 \gamma$  in state b.

Whereas the probability to be in this state decays with twice that because the probability is squared. So you find that pretty much in any quantum mechanically corrects derivation, which you do about the decay of population coherences. Other questions?

So we have done two things. We have done the very, very simple derivation using the beam splitter model where you may not even notice where I did the Markov approximation because I jumped from beam splitter to beam splitter and left all the correlations behind. Here in the most general calculation, you have seen exactly where it enters, but maybe now you have the full forest in front of you, and you don't recognize the trees anymore. So let me wrap up this lecture by now focusing on the system we want to discuss further on, namely a two-level system interacting with a vacuum through spontaneous emission.

But I also want to make some generalizations. I want to give you some generalizations about what kind of environments are possible in quantum physics. Let me just see how I do that. So this part I actually owe to Professor [? Ikschuan ?] who wonderfully compiled that.

So what I want to do now is, I want to call your attention to the operator form which, is rather unique. Remember when we did second order perturbation theory, we had sort of the commutator of  $v$  with the commutator of  $v$  and  $\rho$ . This came from iterating the exact equation of motion for the density matrix. And you want to specialize that now to Jaynes-Cummings model. I mean, in the end, at least in this course, we always come back to the Jaynes-Cummings model because it captures a lot of what we want to explore.

So the Jaynes-Cummings model in the rotating wave approximation is very simple. It raises the atom from ground to excited state and destroys a photon, or it does the opposite. So this is our simple interaction between our system, the two-level atom

and our reservoir which is just the vacuum of all the modes.

And you, again, recognize what I said in general. You usually always find it by linear form, an operator which acts on the modes, on the reservoir on the vacuum, and an operator which acts on the vacuum.

Now, we want to make the explicit assumption that the initial state of the reservoir is the vacuum state. It's empty. And I want to show you what is the structure of the operators we obtain. And so if you put  $v$  into here, you have first the commutator with  $\rho$ , which I write down here, and then we have to take another commutator with  $v$ . And the result of that is the following, that when it comes to relaxation pauses, based on the general structure of the time evolution of quantum mechanics, we have this double commutator.

And the operator which, couples our system to the environment is erasing and lower an operator. I mean, the atom because it interacts with the environment either absorbs the photon or emits the photon. But those operators,  $\sigma_+$  and  $\sigma_-$ , appear now always as products because we have two occurrences of the interaction,  $v$ . But if you look at the double commutator structure, the operator  $\sigma_+ \sigma_-$  appears. This is just the general structure of this double commutator appears to the left side of the atomic density matrix to the right side of the atomic density matrix and then there is the coarse term where the atomic density matrix is in the middle of the two.

So this is actually something which is very general and very important in the theory of open quantum system. What I'm discussing with you now is this famous Lindblad form. And the story goes like that. You want to know what are possible environments, not just empty vacuum. You can have fluctuating fields. You can have, you name it. But if you are saying that your environment interacts with your system through an operator, and our operator is now the operator  $\sigma_-$ , which is a spontaneous emission, you need heavy  $t$ .

The mathematical structure of a valid quantum field remains that if your system interacts with an environment by emitting a photon  $\sigma_-$ , this is now the

structure of the master equation. This is the structure of the time evolution of the density matrix. So the operator sigma minus and its and its Hermitian conjugate sigma plus, all of this have to appear in this combination. Yes, Collin.

**AUDIENCE:** This is still in my interaction picture. Right? There's no dynamical phase evolution that we put back in there.

**PROFESSOR:** Yeah. OK. What we do in general if the system is driven by a laser beam, for instance, Rabi oscillation, we simply add up the dynamics of the Rabi oscillation of the unitary time evolution to the time evolution done by the reservoir.

**AUDIENCE:** So this form is always in the interaction picture?

**PROFESSOR:** Well, this is, you would say, this form is what is the relaxation provided by the environment. And if you drive the system in addition with the coherent field, unitary time evolution, you would add it to it. So in other words, what I'm telling you here is that this is the general structure, and if you have a system which interacts with an environment in five different ways, with a dipole wand, with a magnetic moment and such, you have maybe five interaction terms and then you have to perform the sum over five operators, and here one of them is a sigma minus operator.

So in other words, if you want to know what is the whole world of possibilities for quantum system to relax and dissipate with an environment, you can pretty much take any operator which acts on your system, but then put it into this so-called Lindblad form and you have a possible environment. And I mentioned I think last week that people are now in our field actively working on environmental engineering. They want to expose a system to an artificial environment and hope the system is not relaxing, let's say, to a broken ground state, but to a fancy correlated state.

So what this Lindblad form, if the operators appears in this way, what it insures is the following. Just imagine if we have an equation, a derivative of the density matrix, which depends on the density matrix, you could write down a differential equation and say is it possible? Well, it has to be consistent with quantum physics. You have

certain requirements.

One requirement is that  $\rho$ , the density matrix, always has to be the density matrix. The trace equal 1 has to be conserved. A density matrix always must have non-negative eigenvalues, otherwise, what you write down, it might be a nice differential equation, but quantum mechanically, it's nonsense.

But now there is one more thing, which is also necessary. This time evolution of the system's density matrix must come from a unitary time evolution of a bigger system. So you must be able to extend your system into a bigger system, which is now the environment, and this whole system must follow a Schrodinger equation with a Hamilton as a unitary time evolution.

And this is where it's restrictive. You cannot just write down a differential equation and hope that this will fill some requirement, and what people have shown is under very general assumptions, it is the Lindblad form which allows for it. So some operator always has to appear in this form.

So often in this Lindblad form, you have an operator, which is called a jump operator, which is responsible for the measurement, which the environment does on your system. The jump operator is here, the operator which takes the atom from the excited to the ground state. With that you need a photon, and the photon can be measured.

So often you can describe a system by a jump operator, and if the jump operator is put into this Lindblad form, then you have a valid master equation for your system. So let me wrap up.

If you take now the definition of the raising and lowering operator, and you take the form I showed you, the Lindblad form, you'll find now this differential equation for your two-level system. And this is one part of the Optical Bloch Equation. Now, coming to Collin's question, if you include the time evolution of the classical field, this is a coherent evolution of the Bloch vector, which I showed at the beginning of the class, and we add this and what I wrote down at the beginning of the class.



Then we find the famous Optical Bloch Equations in the Jaynes-Cummings model.

So these are now the Optical Bloch Equations, and I hope you enjoy now after this complicated discussion, how simple they are. And it is this simple set of equations, which will be used in the rest of the course to describe the time evolution of the system.

Just because I did some generalizations about the Lindblad equation, I copied that into the lecture notes from Wikipedia, and probably now you sort of understand what is the most general Lindblad equation. It has a Hamiltonian part and then it has jump operators like you sigma minus operator, but it has to come in the form that the jump operator and its complex conjugate, emission. conjugate, is on the left side, on the right side, and left and right of your density matrix. So this is the generalization I've mentioned.

Yeah. with that I think with that we've derived the master equation, and on Wednesday, we will look at rather simple solutions, transient and steady-state solution of the Optical Bloch Equation. Any questions?

One reminder about the schedule, this week we have a lecture on Friday because I will not be on town next week on Wednesday. And of course, you know today in a week, next week on Monday. It's [INAUDIBLE] day. So we have three classes this week, no class the following week, and then the normal schedule for the rest of the semester.