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**PROFESSOR:** My first question is, do you have any questions about the last class, about the introduction, about the course overview, the syllabus, any requirements of the course, formalities? It should be pretty clear. So today, we start from first principles.

I want to give you a rigorous derivation of what this course is based, namely, the Hamiltonian to describe the interaction between atoms and light. I mentioned to you last week that atomic physics can be defined about everything that is interesting about the building blocks of nature, atoms interacting with light electromagnetic fields, Coulomb fields, and today, we'll talk about the first principle, Hamiltonian.

The reference for that, and I'm closely following that derivation, is the book *Atom-Photon Interaction*, the green book by Cohen-Tannoudji, Grynberg, and others, and it's taken from the appendix. And if you flip through the appendix, you will find out that there are about 100 equations. Today, we'll discuss every single of them.

However, the good news is the result in the end is simple and intuitive. In the end, we have what we want, that the electromagnetic field couples different energy levels, and then we can play around with quantum case, with laser cooling. All we need is a coupling matrix element. And for most of the course, I will not even elaborate what this matrix element is about. But here, we derive it from first principles.

The result we will use most often is actually the electric dipole approximation, which I'm sure you have already seen many, many times. So in that sense, I sometimes have second thoughts. Should I dedicate a whole lecture to derive something you know already? On the other hand, this is sort of the meat of atomic physics, and I want to go as deep as possible into it, that whenever you wonder what form of other

interactions exist, you have a reference to look it up. And so to some extent, I want to also encourage you to read more about the fundamental nature of how light and atoms interact. I will give you another reference a little bit later on.

But as a motivation, I want to tell you what I learned when I prepared this lecture, and I hope that there is something that you will learn from this treatment. One is a rigorous separation of local fields from radiation fields. Let me just write it down, rigorous separation of local Coulomb fields from radiation fields. So to clearly separate the terms in the Hamiltonian, let me explain that.

When we introduce electromagnetic fields into our world, we would say we introduce them by saying particles have charge. But some parts of the charged particle physics is something we want to include in the description of atoms, and that's the Coulomb field of protons. The structure of atoms is electromagnetic, but when we now introduce electromagnetic field as a new degree of freedom-- photons can be admitted, absorbed, and such-- we only want to talk about the electric field which belongs to the photons and not the electric field which belongs to the atoms.

And you may wonder, can you distinguish between two kinds of electric fields? The answer is yes, and I want to show you how mathematically you distinguish when you introduce electromagnetic fields between the fields which stay with the atom, which don't have an extra degree of freedom, and if you do canonical quantization, they don't enter as an extra variable, and the fields which are the photons, and these are the extra objects we have to consider.

What is also related is when you do second quantization, if you wanted, I could quantize electromagnetic field for you in five minutes. Just say it's harmonic oscillator. You're familiar with harmonic oscillator, and let's just do it. Yes, this is correct, and I will remind you of that a little bit later, but there is one non-trivial aspect if you want to do quantization of a field theory, you have to make sure that before you quantize it, you know which degrees of freedom are really independent. So the question is, how many independent fields does the electromagnetic field have? The naive answer is three electric field components, three magnetic field

components, that makes six. But of course they're not independent, and we will discuss that.

So we'll spend some time in the identification of truly independent field components, or degrees of freedom of the electromagnetic field. So all that is an excursion into classical physics because it is just reformulating classical electrodynamics to be ready for quantization. Actually, all the work is done to discuss and derive the appropriate classical description. That means to eliminate all redundant variables, and then at the end, have variables which, yes, will look like a harmonic oscillator. And once we have reduced it to degrees of freedom which look like a harmonic oscillator, it is very straightforward to do the field quantization following the recipe of the harmonic oscillator.

There's one more highlight, if I want to say so, and this is a truly rigorous derivation of the electric dipole approximation. In most quantum mechanics textbooks, you do a dipole approximation and you wind up with the electric dipole Hamiltonian, which I have already put up there, but you do actually unnecessary approximations in terms of neglecting an a squared term, and there are often confusing discussions what happens with the a squared term.

I want to show you a more rigorous deviation of the electric dipole approximation, which is including the quadratic term. That actually means-- that's the take home message-- the electric dipole approximation is actually better than many of you have thought so far.

Since I don't want to write out 100 equations, I copied them and summarized them out of the book. You'll find them all, as I mentioned before, in the appendix of *Atom-Photon Interaction*. But I sort of want to walk you through. What I'm aiming at is that you have sort of a complete overview how is everything coming together, and if it's just mathematical, well, mathematics is always exact. I can go faster, but I really want to highlight here every single physical concept and approximation we are doing.

When we start out with Maxwell's equations, we have actually six components of the

electromagnetic field, and we will see in a moment that this is redundant. What will turn out to be very important to separate the local fields from the radiation field is a Fourier transformation, so we introduce an expansion into plane waves. And what then happens, of course, is the derivative operator becomes the  $k$  vector.

So therefore, when you have Maxwell's equations with the curl,  $\nabla \times \mathbf{B}$ , it turns into  $k \times \mathbf{B}$ . This is now important because we have now separate equations for the component of the electric and magnetic field, which is parallel to  $k$ , projected onto  $k$ , and of course, the cross product takes out the component which is perpendicular to  $k$ . So that's something which is sort of nice, that the Fourier transform it allows us to separate the fields into longitudinal fields, where the vector of the Fourier component is parallel to the  $k$  vector, and into transverse components.

**AUDIENCE:** [INAUDIBLE]?

**PROFESSOR:** The difference between curly and non-curly vector is one is a Fourier transform, the other one is other spatial components. I have to actually say the book *API* tries to be super accurate in choosing [INAUDIBLE]. Claude Cohen-Tannoudji is really a master of elegance and perfection, so they don't want to use the same [INAUDIBLE] for a spatial component and for the Fourier component.

On the other hand, I have to tell you one thing. For the next hour, use a little bit your intuition. If you see  $e$ , it means electric field. Whether it is curly, whether it's italic, whether it's bold face, don't let those differences clutter your view. So I would say don't ask me too often, what exactly does it mean now because you have gone from curly to non-curly? It is explained in the appendix. I will tell you everything which is necessary, but these are subtleties which are more to present the mathematics in a more rigorous way.

We have the spatial Fourier components, and as I mentioned, what you can do only because of the Fourier transform, you have now the distinction, a rigorous separation between the parallel and the longitudinal fields. So of course, you can also now get back what is the longitudinal field, not in Fourier space but in position

space, by just taking the longitudinal or transverse components of the Fourier transform and transforming them back.

Why I'm elaborating on that is we will see in the next few minutes that the transverse field is the field which propagates. The longitudinal field is the Coulomb field. It belongs to the atoms and will not become a new degree of freedom. And we see that immediately in the next equations.

When we look now at Maxwell's equations, we find that the transverse and longitudinal fields decouple. We have two kinds of equations which are completely decoupled. The longitudinal electric field is the Coulomb field associated with a charge density, whereas the transverse fields are creating themselves, and this is radiation, how the electric and magnetic fields create themselves as they propagate.

Now, the fact that the parallel electric field can be expressed by the charge density, or it's expressed by the momentary position of charges, and this means immediately that the longitudinal field is not an independent variable or an extra degree of freedom. Canonical, of course, with how we describe particles by their position, and here you see explicitly that the longitudinal field which we defined the way we did it, that this part of the field only depends on the momentarily electric field, the Coulomb field associated with the charges.

The next thing, of course, is to define the vector potential and the scalar potential. By using the standard substitution, we can now express electric and magnetic fields by a scalar potential and by a vector potential. We continue to have the separation between longitudinal and transverse fields, namely the transverse electromagnetic fields depend only on the transverse component of the vector potential. And the transverse component is gauge invariant. In other words, you know there are different gauges we can consider when we introduce electromagnetic potentials, but the choice of gauge does not affect the transverse vector potential.

Now we want to pick our gauge and repeat the gauge which is most convenient. This is typically, for low energy quantum physics, for describing atoms and radiation, the Coulomb gauge. So we have the freedom of gauge, and the Coulomb gauge is

written by the divergence of the vector potential is 0. If you now think about Fourier transform, this means  $\mathbf{k} \cdot \mathbf{A}$  is 0, and that means the vector potential does not have any longitudinal component.

So with that, we have now reduced equations for six variables, three electric fields, three magnetic field components. By introducing vector and scalar potential, you go to four. We've eliminated one more with the Coulomb gauge, so we have three. The scalar potential is the Coulomb potential. It's not an extra degree of freedom.

And what is left are the two field components of the transverse vector potential. And these are now, as we will see, the independent variables of the radiation field. So in other words, when we talk about the electromagnetic field in the remainder of the course, the part of the electromagnetic field we are interested in are the fields generated by the transverse vector potential.

Now, what is next is to go into normal modes. The reason is that we want to identify each normal mode of the electromagnetic field as an independent harmonic oscillator. Now, I'm showing you those equations, but you sort of have to sit back because now there will be alphas,  $\alpha$ 's, lower  $\alpha$ 's, upper  $A$ 's. They all mean pretty much the same. In one case, you are normalized, maybe something has to be factored out or not. The derivation in the appendix of *API* is exactly [INAUDIBLE] distinction, but let me just give you a flyover here.

And that is we are now taking the transverse components from the electromagnetic field, and we define new parameters, which are the normal modes. What we actually want to see in a moment is that those normal mode variables are harmonic oscillators. Those normal modes can be defined by the original field, which are transverse, but since we have expressed the transverse field by the transverse vector potential, they also simply can be expressed by the transverse vector potential. So to remind you, we've done a Fourier transform in the spatial coordinate, but not yet in the temporal coordinate.

What we obtain now is an equation of motion for the normal mode variables. And here, I have reminded you that the transverse vector potential, how it can be

expressed by the normal mode variables. In other words, yes, it looks complicated and a little bit messy, but it's classical physics, and all we have done is introduce normal modes in a just more complicated notation, as you may have seen it in 803 or some other course where for the first time you talked about normal modes of a pendulum or a chain of springs. So we have identified normal modes by the equation above, and then it's a mathematical identity that our radiation field, our transverse vector potential, can simply be expanded into normal modes.

So the equation of motion for the transverse fields involves of course, the transverse vector potential. I should have scrolled back. Here we have our equations for the transverse fields. This is what we want to describe, and if we express everything in terms of the vector potential, then we have an equation which involves the transverse vector potential, but in addition, because it's a differential equation, the first and second derivative.

That means that if you have a second order differential equation, which is the full description of the electromagnetic field, then you know that the solution of the second order equation needs as an initial condition the field itself and the first derivative. Therefore, our classical fields are determined by a perpendicular and its derivative at the initial time.

So we need a and we need a dot, and they are coupled, but what we are doing right now is-- and this is the idea behind the normal modes-- the normal modes coordinates combine a and a dot. It's the same when you have a harmonic oscillator and you want to introduce coordinates for the normal mode, they are a combination of position and momentum, and then they are decoupled. But position and momentum always couple or oscillate back and forth, and the same happens here between a and a dot, and the normal mode coordinates are the linear combination.

In other words, the equation of motion coupled source and normal mode means that we have introduced the decoupled normal modes. And whenever, in classical physics, we have normal modes, normal modes means that time dependence is  $e^{i\omega t}$ , then we have distilled the problem of coupled components to

decoupled harmonic oscillators. So at this point, each mode acts as an independent harmonic oscillator.

So what I wanted to show you here is clearly, as a flyover, and as an appetizer to read more in the book, that we can start with the electromagnetic field components. We haven't just assumed we have now empty space and only [INAUDIBLE] radiation. We have rigorously separated the electromagnetic field in what belongs to the atoms and what belongs to radiation, and the trick was a spatial Fourier component.

At that moment, we had a description in terms of the transverse vector potential, and by using purely classical physics, by combining the vector potential and its derivative, we found normal modes, and they are now completely independent harmonic oscillators. This approach, and I wanted to show that to you, involved a lot of notation--  $a$ 's, little alphas, normal modes, and such.

Let me just simply show you another much, much shorter pathway how you see that everything looks and smells like a harmonic oscillator, and then we'll do the last step, which is rather straightforward. We quantize the electromagnetic field. I could quantize it right now, but we should take a short break from the appendix in *Atom-Photon Interaction* and do a more intuitive shortcut to the same physics. Any questions at this point? If you feel it's confusing, remind yourself this is just classical physics. We are just rewriting Maxwell's equations in new variables.

So what we do now is, as I said, let's repeat some of this deviation by focusing on energy. If you write down energy for our systems of atoms, photons, and Coulomb fields, we have particles  $\alpha$  with mass  $m_\alpha$  and velocity  $v_\alpha$ . This is our kinetic energy. And then we use the well known expression for the electromagnetic field, which is the spatial integral over  $E^2$  plus  $B^2$ . This integral over the electromagnetic field energy density can actually be nicely separated into integral over the longitudinal field which we have introduced and a second integral, which involves the transverse field. Of course, the magnetic field is only transverse because the divergence of the magnetic field is 0, and that means the magnetic

field does not have any longitudinal component.

So this part here, as we actually have shown, is given by simply the Coulomb energy where we have charge density at position  $r$  and  $r$  prime interacting with the Coulomb interaction. Eventually, if you want to treat that further, this Coulomb energy should be split into some divergent self energy, which is the energy of the electron interacting with itself, and people know how to deal with that, and the interaction energy, for example, between the proton and the electron, which is responsible for the atomic structure.

So in other words, we know how to deal with that. This actually becomes atomic structure. This is in contrast to this part here, which is the classical energy of the radiation field, which is sort of the transverse part of the energy.

To describe now radiation, we have to focus on this transverse part, and we are now expressing the transverse part by the vector potential and its derivatives. So we introduce the vector potential as before, and I don't want to go through the re-derivation. I just want to show you how the total energy now appears in terms of the vector potential.

And what we need for that is, of course, the vector potential and its derivative. The vector potential depends on polarization, depends on the Fourier component, and we call this derivative of the vector potential the conjugate momentum. So now focusing on energy, we, of course, reduce everything as before to the vector potential and its derivative.

So we sum over polarization. We integrate over all Fourier components. and the integral involves now the vector potential, the square of it, or in complex notation, the complex conjugate, depends on polarization. We have  $c^2 k^2$ . This part, of course, comes from the magnetic field, which depends on the vector potential. It's a spatial derivative, the curl, and this gives the  $k^2$ , whereas the part of the electromagnetic energy, which is related to the electric field, the electric field is a temporal derivative of the vector potential, and therefore, it involves the temporal derivatives which are now the canonical momenta.

So this equation should really remind you now-- it's an energy equation-- of the energy of a harmonic oscillator because the energy is now a sum over all oscillators, but this is sort of  $x$  squared, the amplitude of the oscillator squared, this is the potential energy. And here we have the derivative of  $x$ , the velocity or momentum, and this should remind you of the kinetic energy of a harmonic oscillator. In other words, this should tell you that by focusing on the transverse component of the vector potential, it's about lots of summation--  $k$  summation, polarization summation. We in the end find that each such mode is a harmonic oscillator. Any questions?

Let me just show you now, and it may help you to go through the previous derivation, let me now introduce what I did before, the normal modes, and then show you how the energy looks like defined in normal modes. So we have these normal mode variables, which are defined as a superposition of  $A$  and  $\dot{A}$ . And I'm not telling you here with all the indices whether it's the polarization component, the  $k$  component. I just want to tell you the structure. The normal mode is a superposition of  $\dot{A}$  and  $A$ .

And with that, the energy of the electromagnetic field can be written-- well, we always have to sum over Fourier components. We always have to sum over the polarization. But then we have something which is very, very simple and intuitive. It's just the square of  $\alpha$  with the correct polarization, the pre-factor is  $\hbar \omega$  over 2.

It looks like quantum mechanics,  $\hbar \omega$  times  $A^\dagger A + 1/2$ , but this is purely classical. There is no quantization, there is no operator. We've simply defined something new, namely  $\alpha$  in terms of the vector potential. In that sense, if you're going to go back to the derivation, the  $\alpha$ s are nothing else than some elaborate combination of the transverse electric and magnetic field. Colin?

**AUDIENCE:** Where does the  $\hbar$  come in? Is that the coefficient of  $\alpha$ ?

**PROFESSOR:**  $\hbar$  only enters through the constant in the definition of the normal mode

parameter. So actually, I have introduced  $\hbar$  by choosing this parameter wisely in such a way that it connects with quantum mechanics. But it's a completely arbitrary introduction here. I could have said  $\hbar$  to 1.

I hope you enjoyed, or at least did not dislike this excursion into classical physics. We have now two equations for describing the energy of a harmonic oscillator. So both of these equations look like a harmonic oscillator, but if something looks like a harmonic oscillator, it is a harmonic oscillator.

And I think I should always remind you that so far, everything has been purely classical. And also, let me write down that  $\hbar$  enters solely through the constant in the definition of the normal mode parameter,  $\alpha$ .

If you want to know more about it, there is a second reference. I will actually show you the cover page in a few moments. But there is a second book by Claude Cohen-Tannoudji and collaborators, not *Atom-Photon Interaction*, but called *Photons and Atoms*. It's a whole book on rigorously defining QED. So a whole book has been written about the subject of this lecture, and on page 27, you can read more about that.

So enough of classical physics. All we have done so far-- I'm sorry for repeating it, but I think I can't repeat it too often with so many complicated equations on the screen-- all we have done so far is we have rewritten Maxwell's equations. We have rewritten Maxwell's equations in Fourier space, with the vector potential, eventually with normal modes. That's all we have done.

But now what we are doing is we do the step in quantum mechanics which you have already seen a few times, and this is you have written equations in such a way that they look like harmonic oscillators, and then you postulate that the classical quantities become operators. The transverse vector potential and the conjugate canonical momentum fulfill now commutators, and we use those commutators to define them as operators.

Now, it looks particularly easy when we use normal mode operators because the

normal mode operators after quantization become our  $a$ 's and  $a^\dagger$ 's. That was one of the things I wanted to show you, that you can go through everything introducing classical normal modes, and then the normal modes turn into operators, and now we have all your creation and annihilation operators.

In other words, quantization cannot be rigorously proven. I mean, you cannot prove quantum theory from first principles. You can have a mathematical framework and check it against nature. What we have done here is to formulate the quantum theory. At this point, we've made a postulate that we have operators which fulfill a commutation relation. And it is now your choice if you want to formulate postulate the commutator for the transverse vector potential and its conjugate momentum, or if you immediately want to jump at the normal modes, and that would mean you have the commutator for  $a$  and  $a^\dagger$ .

We are almost done with that. We started with the electromagnetic fields. We went through transverse fields, vector potential, normal modes, now we quantize. But now all the equations, all the substitutions we have made, we can go backward. Therefore, we can now express the transverse vector potential, the electric and the magnetic field, by the normal modes, or that would mean in quantum mechanics, after quantization, we can express them by the  $a$ 's and  $a^\dagger$ 's, and this is how we define the operator of the electric, the magnetic field, and the vector potential.

So now, our fully quantized theory has operators,  $a$  and  $a^\dagger$ , or if you want now, the first defined operators of vector potential, electric and magnetic field. And just to remind you, since atomic physics is between fields and particles, we have exactly the standard definition of the particle operators. Each particle is described by its momentum and its position. So with that, we have our Hamiltonian, and a lot of what we discuss in this course are understanding this Hamiltonian, understanding its solution, understanding what is the physics described by that.

So this Hamiltonian, this is the kinetic energy. The canonical momentum minus the vector potential is the mechanical velocity. This is kinetic energy. And we have separated the energy of the electromagnetic field into the Coulomb energy, which is

written again here, and the radiation field.

There is now one more term which we need later on, which is this one here. And at this point, I would say-- I'm just heuristically adding it by hand-- classical we don't have spin, but our particles have spin, and now we need [INAUDIBLE] coupling, how does the spin couple to the rest of the world?

Well, the spin, if you multiply it with the g factor, the Bohr magneton has a magnetic moment, and what we simply add here is  $\mu \cdot p$ , be the interaction of the magnetic moment with the magnetic field. So in that sense, we have been classical all the way. Now at the end we said, we need the spin. Let's put it on by simply taking an interaction, which is  $\mu \cdot p$ .

If you don't like that because I try to be very fundamental today and go from first principles, for electrons, you can get that by taking the non-relativistic unit of the Dirac equation. So if you have electrons, you can start with the Dirac equation and do the Pauli approximation, which is the non-relativistic limit of the Dirac equation. Any questions? So these are all the terms in the Hamiltonian.

I mentioned, but I don't want to dwell on, that the Coulomb energy has to be separated into a Coulomb self energy and an interaction energy between the charged particles, but all of that becomes just one term in our Hamiltonian. This is the atomic structure. When we assume that atoms have energy levels, all the energy is included in that, so we will not discuss any further the Coulomb energy. We will simply assume we have an atom which has certain energy levels, and that includes all the Coulomb terms.

But we will talk a lot about the Hamiltonian for the radiation field. The Hamiltonian for the radiation field can be conveniently written in  $a$  and  $a^\dagger$ , but I have a few equations up there I rigorously defined for you operators  $E$  and  $B$  in terms of  $a$  and  $a^\dagger$ s, and those two equations are identical. So this looks very quantum, this looks very classical, but if you interpret the electric and magnetic field as operators, we have identical equations for the Hamiltonian of the radiation field.

What we want to study are interactions between light and atoms, and of course, this comes from the vector potential. And in particular, when we square it out, the canonical momentum of the atoms times the vector potential has this cross term,  $\mathbf{p} \cdot \mathbf{a}$ . So let me know just take this Hamiltonian and write it in the way how we will need it for this course.

We want to take the Hamiltonian, and I will often refer to that. We want to take the Hamiltonian and split it into three parts, the atoms, the radiation field, and the interaction between the two. The Hamiltonian for the particles,  $H_P$ , has the momentum squared, and it has the part of the electromagnetic fields which are longitudinal, the energy of which can be described by a Coulomb integral. We have already discussed the radiation field. The radiation field was nothing else than  $\hbar \omega$ ,  $a$ , but the new part which we need now is the interaction term.

I want to show you now, or remind you by just summarizing the term, that the interaction part has actually three different terms. The first one is the cross term between  $\mathbf{p}$  and  $\mathbf{A}$ . The second one, when we had  $\mathbf{p} - \mathbf{A}$  and squared it, is the  $\mathbf{A}^2$  term. And the third one is the interaction of the spin with the magnetic component of the radiation field. This is the  $\boldsymbol{\mu} \cdot \mathbf{B}$  interaction. I don't think it's an exaggeration when I tell you that with those three equations, you can understand all of atomic physics.

This term here that's important is second order in  $\mathbf{A}$ , which of course also means it will be very important for very strong laser pulses, but there's more to be said about it. If you're interested in this subject, and I couldn't do full justice to it, you may want to look at this book from Claude Cohen-Tannoudji, Jacques Dupont-Roc, and Gilbert Grynberg about photons and atoms. The whole book is dedicated how to describe the interaction of atoms with electromagnetic fields.

In this book, they go through different ways of formulating the electromagnetic field classically. They use Lagrangian formalism for the electromagnetic field and show the Euler-Lagrange equation or Maxwell's equation. There are lots of different approaches, and also, one can say what we did here, which is the simplest way of

describing light-atom interaction, will work in the Coulomb gauge, which is not the gauge you would choose when you want to describe relativistic physics. There is the other gauge, the Lawrence gauge.

It makes the field quantization much more difficult, but when you ever wondered, why is the quantization done in the way I did, and you think something I did was arbitrary-- why did I pick the Coulomb gauge? Read this book. There are hundreds of pages which explain it to you. And I'm not joking here, it's wonderful reading. I got the book just looked up a few things and I almost got hooked on it and read more and more. It's a fascinating story how deeply people have thought about how to describe this aspect of the course and how profound the thoughts are.

What this book will emphasize-- and this is what I want to tell you in closing about the quantization of the electromagnetic field-- is the following. We were really working hard with classical field equations to describe the electromagnetic field classically by completely eliminating redundant variables. We started with six components of the electric and magnetic field and reduced it to two components, the two components of the transverse vector potentials. And as those masters say, as a result, the field can be quantized with a great economy in the formalism.

If you want to have more symmetric formulations, where you don't eliminate the variables using the Coulomb gauge, the problem is you have variables, which are not independent. And now I think if you quantize, you have to formulate auxiliary conditions between the not independent operators. Aesthetically, it may be pleasing because the approach is more symmetric, but mathematically, it's much more involved.

Anyway, if you're interested, this book has a wonderful discussion on the different ways how you can approach interactions of atoms with electromagnetic fields. I hope nobody has a question about that, because I'm not able to explain to you much more than that. Anyway, questions?

We have derived from first principles the Hamiltonian, and the result is something many of you are already familiar with. Let me now spend the next 10 or 15 minutes

on what is called the dipole approximation and the dipole Hamiltonian. We are talking now about one further expansion of our expressions, and this is the multipole expansion.

Let me first give you the simple derivation, which you may have seen, but then you can appreciate the rigorous derivation. Either directly or through the vector potential, we have formulated our theory in terms of the electric and magnetic field as a function of position. But what we now want to exploit is that atoms are tiny, and usually, the electric or magnetic field is not changing over the extent of an atom or molecule. In other words, what I'm telling you is the relevant frequencies, the relevant components of the electric and magnetic fields will, of course, be the components which are in resonance with the atoms and molecules.

What I'm telling you is that in many, many situations, the wavelengths of the relevant modes of the electromagnetic field are much, much longer than the size of an atom. So therefore, if we pinpoint our atom at the origin, we may be able to neglect the spatial dependence of the electromagnetic fields and replace it by electric and magnetic fields at the origin. Or, if you want to go higher in a Taylor expansion, we use gradients of it. We've talked more about that, actually, in 8.421, when we talked about multipole transitions, but here I want to only focus on the lowest order, which is called the electric dipole approximation, because this is the most important case which is used in atomic physics.

So for the electric dipole approximation, we make actually several assumptions. One is we neglect the quadratic term, the  $a^2$  term. And now if you look at the  $\mathbf{A} \cdot \mathbf{p}$  term, and we are looking for matrix elements between two atomic levels, we expand the vector potential into plane waves. Well, it's called the electric dipole approximation, so maybe we want to express things by the electric field, which is the derivative of the vector potential.

So with that, we have now described things by the electric field. And what is needed now is if we assume that  $kr$  equals 0, or  $r$  over  $\lambda$  is very, very small, the exponential, the plane wave vector, is approximated by 1, and we are simply left in

the electric dipole approximation with the  $a$  element of the momentum operator.

We can simplify it further by writing the momentum operator as the commutator between the position operator and the Hamiltonian. If you take the commutator between  $r$  or  $x$  and  $p$  squared, you'd simply get  $p$  and  $p$  factors, so that's what we are using. And now the matrix element becomes-- so what we have here now is  $rH$  minus  $Hr$ , but  $H$  acting on 1, because 1 is an eigenstate, gives the energy,  $E_1$ . And for the flipped part, we have  $Hr$ , and now  $H$  acting on state two gives us the energy,  $E_2$ .

So therefore, by taking care of the Hamiltonian, which gives us, depending on which side  $H$  appears,  $E_1$  or  $E_2$ , we have reduced it to now a position matrix element. And what we have here is, of course, the transition frequency between the levels one and two. I know you have seen it, but if I can derive it in two minutes, it's maybe worth the exercise.

What I've shown you is that the momentum matrix element can be replaced by the position matrix element multiplied by the resonance frequency. But now we want to put things together, and what I want to point out for you is that we have an  $\omega$  here, which is the frequency of the electromagnetic field, and we have an  $\omega_{12}$  here, which is the transition energy.

So therefore, if I put it together, I started with a  $p$  dot  $a$  Hamiltonian, but the  $p$  matrix element became an  $r$  matrix element. The  $a$  was expressed by the electric field. So therefore, we have now the electric field times the position matrix element, or the dipole matrix element, if you multiply the position of the electron with the charge  $q$ . But in addition, we have this prefactor. So this becomes the dipole Hamiltonian  $E \cdot d$  when we replace that by 1, assuming that we are interested anyway only in interactions with atoms where the radiation field is near resonance.

So it seems that we have actually made three assumptions to derive the electric dipole approximation. One was the long wavelengths approximation. The second one was that we are near resonance, and the third one was that we neglected the quadratic term in the light-atom interaction.

There are often people who are wondering about it. When I said we are only interested when the atoms interact with electromagnetic radiation near resonance, that's OK, but atomic physics is the area of physics with the highest precision. We can make measurements with  $10^{-16}$  and  $10^{-17}$  precision. So therefore, if you have an electromagnetic field which resonates with atoms-- the typical frequency of visible light is  $10^{14}$  Hertz-- and you're just one megahertz away from the resonance, it's  $10^{-8}$ .

So the question is, would we observe a correction to the dipole approximation if  $\omega^2$  is not exactly  $\omega$ ? Well, the answer is no because, as I want to show you now in the next few minutes, and this is also the end of our flyover over the appendix of atom-photon interaction, the last two assumptions are not really necessary. In other words, we go back to atom-photon interaction, to this long appendix on the derivation of the QED Hamiltonian.

We go back to our fundamental Hamiltonian, and we want to do one approximation and this is the only approximation which is needed. It is the dipole approximation, and that means that the vector potential-- of course, we only need the transverse part of it-- is replaced by its value at the origin. That's the only approximation.

So here is our Hamiltonian in its full beauty, and the only thing we have done is we've put in the origin here. For later convenience, we introduce the dipole operator, which is the position operator of particle alpha times the charge of particle alpha. And then we do simply a transformation. So this is a unitary transformation, and we transform from the original Hamiltonian, which includes the  $\mathbf{p} \cdot \mathbf{a}$  term and the  $\mathbf{a}^2$  term to a transformed Hamiltonian,  $H'$ .

And this transformed Hamiltonian has now the electric dipole approximation. It's sort of written in polarization Fourier space, but this is the electric field, and  $\mathbf{a}$ , a dagger gives position. I'm sorry. These are normal modes. It's a dipole operator times the electric field, and the  $\mathbf{A}^2$  term has disappeared. So after the transformation, we have no quadratic term, and all we are left is this is nothing else than  $\mathbf{d} \cdot \mathbf{e}$ , the dipole interaction. And there is no prefactor  $\omega^2$  over  $\omega$ . This is a

rigorous unitary transformation to another basis.

In full disclosure, it involves a dipolar self energy, which is sort of a constant energy. And if you look at the transformed velocities, the transformed velocities  $v$  prime are now identical to the original canonical momentum. So after that, what was the canonical momentum in the original Hamiltonian,  $H$ , becomes now velocity times mass. So  $p$  becomes now the mechanical momentum. And therefore,  $p$  squared is the kinetic energy.

Anyway, I think this is important that we have this much more rigorous deviation of the dipole approximation. I don't want to go through that. I just want to make sure you have seen it. If you want to do it really rigorously, then you have to distinguish between the electric field  $d$  and  $E$ . So you have an electric field here, and here you have the polarization. It's pretty much the same as in classical  $e$  and  $m$ .

At the end of the day, we have now, for the prime operator, for the Hamiltonian after the canonical transformation, after the unitary transformation, the only interaction term which remains is the electric dipole interaction between the dipole and this field  $d$ , but this field  $d$  prime corresponds to the original a transverse electric field. I'm not going to explain the difference between  $E$ ,  $E$  prime,  $d$ , and  $d$  prime. It can be done rigorously.

But the take home message is-- and with that, I want to conclude-- that for most of this course, the interaction of the electromagnetic field with our atoms is described by this term. We take the transverse electric field. This is our radiation field, and it couples to the operator of the atomic dipole moment. I hope you got the take home message that whenever you have any doubts about any part of this deviation, or if you have any doubts whether it's rigorous or not, you know now where to look it up and learn everything about it. Any questions?

OK, then that's it. We'll probably post the first homework assignment tomorrow-- I will meet with the TAs-- and it will be due in about a week, but you will see all of that on the website and hear from me on Wednesday. Have a good afternoon.