Identical particles & second quantization.

For a system of N identical particles, all physical observables are invariant under an exchange of two particles.

If $\psi(x_1, \ldots, x_N)$ is the wave function of a system of N identical particles,

$$|\psi(x_1, \ldots, x_N)|^2 = |\psi(x_2, x_1, x_3, \ldots, x_N)|^2$$

$$\Rightarrow \psi(x_1, x_2, \ldots) = e^{i\theta} \psi(x_2, x_1, \ldots, x_N).$$

As 2 exchanges returns the system to its original state

$e^{2i\theta} = 1 \Rightarrow e^{i\theta} = \pm 1$.

(With spin the exchange must involve spin indices too:

$$\Psi(x_1\sigma_1, x_2\sigma_2, \dots) = \pm\Psi(x_2\sigma_2, x_1\sigma_1, \dots)\Big).$$

Symmetry under exchange imposes a restriction on the allowed Hilbert space for the system of identical particles.

For bosons $e^{i\theta} = +1$, and the ~~[illegible]~~ states allowed in the Hilbert space are symmetric under exchange of any two particles

– more precisely are eigenstates of the permutation operator with eigenvalue $+1$.

For fermions $e^{i\theta} = -1$, and the states allowed in the Hilbert space are antisymmetric under exchange of any two particles.

– i.e. are eigenstates of the permutation operator with eigenvalue -1.

Comments:

1. Despite its familiarity, Fermi statistics is from a certain point of view quite wierd. Exchange of two fermions even if they are very far away still changes the wave function (by a phase factor -1) — can be regarded as an "infinitely" non-local "interaction" between 2 identical fermions.

2. For identical particles in 2 spatial dimensions, more exotic possibilities exist where the statistics is fractional (i.e under exchange the phase factor $e^{i\theta} \neq \pm 1$) or even non-abelian (where under exchange, the wave fn changes by multiplication by a unitary matrix & the matrices for different exchanges do not commute). Both these exotic possibilities in $d=2$ are realized

- at least theoretically - in the fractional quantum Hall effect.

For the present stick to bosons / fermions.

Given an arbitrary function $u(x_1, \cdots, x_N)$, can always symmetrize or antisymmetrize it.

$$\Psi(x_1, \cdots, x_N) = \frac{1}{N!} \sum_P u(x_{P_1}, x_{P_2}, \cdots, x_{P_N})$$

(Bose)

$$= \frac{1}{N!} \sum_P (\mathrm{sgn}\, P)\, u(x_{P_1}, x_{P_2}, \cdots, x_{P_N})$$

(Fermi),

where P denotes a permutation of $(1, \cdots, N)$.

$\mathrm{Sgn}(P) = +1$ for even permutation

$= -1$ " odd "

Consider special case where

$$u(x_1, \ldots, x_N) = u_1(x_1)\, u_2(x_2) \ldots u_N(x_N)$$

i.e. is a product of separate functions of each of the N coordinates.

Antisymmetrized wavefunction

$$\psi(x_1 \ldots, x_N) = \frac{1}{N!} \begin{vmatrix} u_1(x_1) & u_1(x_2) & \ldots & u_1(x_N) \\ u_2(x_1) & u_2(x_2) & \ldots & u_2(x_N) \\ & & & \\ u_N(x_1) & u_N(x_2) & \ldots & u_N(x_N) \end{vmatrix}$$

This is known as the Slater determinant.

Similarly the symmetrized wavefunction is

$$\psi(x_1, \ldots x_N) = \frac{1}{N!} \underbrace{\sum_P u_1(x_{P_1})\, u_2(x_{P_2}) \ldots u_N(x_{P_N})}$$

$\underbrace{\quad}$ is sometimes known as the "permanent".

Note: The antisymmetric wavefunction vanishes if $u_k = u_l$ (if $k \neq l$) or $x_i = x_j$ (if $i \neq j$)

This is the Pauli principle for fermions.

More generally consider any complete orthonormal set of single particle states $|\{\alpha\}\rangle$.

~~A basis~~ Many particle states may be constructed as tensor products of these single particle states

$$|\alpha_1 \cdots \alpha_N\rangle \equiv |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes |\alpha_N\rangle .$$

Again we may suitably symmetrize / antisymmetrize these states as appropriate for bosons / fermions respectively.

$$|\{\alpha_1 \cdots \alpha_N\}\rangle = \frac{1}{N!} \sum_P \xi^P |\alpha_{P_1} \alpha_{P_2} \cdots \alpha_{P_N}\rangle$$

$\xi = +1$ for bosons

$\;\;\, = -1$ for fermions.

The states $|\{\alpha_1 \ldots \alpha_N\}\rangle$ form a complete orthonormal basis for the Hilbert space of the many particle system.

Occupation # representation.

The information in $|\{\alpha_1 \ldots \alpha_N\}\rangle$ may also be represented by specifying the # of times n_α any particular α occurs in the state.

Clearly $0 \leq n_\alpha < \infty$

But for fermions since each α occuring in the state cannot be repeated, must have $0 \leq n_\alpha \leq 1$, i.e

$n_\alpha = 0$ or 1 for fermions (Pauli exclusion)

For bosons n_α can be any non-negative integer a priori.

$\therefore$ We may represent the state by the collection $|\{n_\alpha\}\rangle$ of the "occupation #s" n_α.

The occupation # basis is clearly an equivalent complete orthonormal basis.