8.821 String Theory
Fall 2008

8.821 F2008 Problem Set 2 Solutions

This problem set contains a high number density of fermions and their associated minus signs. While everything here appeared to work out, really that only means that I succeeded in making an even number of mistakes, so please treat all formulas here with caution.

I. HOW TO REMEMBER THAT $\mathcal{N}=4$ ACTION

The 10d action that we are going to dimensionally reduce is as follows

$$S_{10d} = -\int d^{10}x \frac{1}{4g_{\rm YM}^2} \mathrm{tr}\left(F_{MN}F^{MN} + 2i\bar{\lambda}\Gamma^M D_M \lambda\right) \tag{1}$$

Here M, N are 10d indices, μ, ν 4d indices, and i, j run over the six dimensional space we are reducing on torii. We will do this problem in various increasingly detailed stages. Depending on how much you like dimensional reduction and spinor representation theory, you can stop reading whenever you want.

1. **No work at all:** This 10d theory has 16 supercharges and no fields higher than spin 1. Its dimensional reduction to four dimensions also has 16 supercharges and no fields higher than spin 1. The only such theory in four dimensions is $\mathcal{N}=4$ SYM, whose Lagrangian is completely determined by its symmetries. Therefore it must work out.

2. **Bosons only:** Just for peace of mind we might as well work out the bosonic sector. Assuming that nothing depends on x^i, we can write down the components of F_{MN}:

$$F_{ij} = -i[A_i, A_j] \qquad F_{\mu i} = \partial_\mu A_i - i[A_\mu, A_i] \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{2}$$

From here, the first kinetic term from (1) becomes

$$F_{MN}F^{MN} = F_{\mu\nu}F^{\mu\nu} + 2(\partial_\mu A_i - i[A_\mu, A_i])(\partial^\mu A^i - i[A^\mu, A^i]) - [A_i, A_j][A^i, A^j] \tag{3}$$

The components A_i are really just scalars in the remaining $4d$ space, so from now on we will call them X_i instead. Noting also that the middle term is really the covariant derivative of a scalar in the adjoint of the gauge group, we obtain

$$F_{MN}F^{MN} = F_{\mu\nu}F^{\mu\nu} + 2(D_\mu X_i)^2 - 2\sum_{i<j}[A_i, A_j]^2 \tag{4}$$

which is indeed the bosonic part of the $\mathcal{N}=4$ 4d action. Surely the fermions will also work out.

3. **Fermions too:** First, note that the 10d Majorana Weyl spinor λ has 16 real degrees of freedom. This is because the 10d Dirac spinor had 32 complex; the Weyl spinor thus has 16 complex, and the Majorana constraint cuts this down to 16 real (recall it is in ten (and two) dimensions that one can have a spinor that is both Majorana and Weyl). The $\mathcal{N}=4$ action has four Weyl spinors, each with two complex (four real) degrees of freedom; so the counting matches. We now need to work out how they reorganize themselves. This is essentially a nightmare of notation, and I encourage you to stop reading now.

We are forced to introduce the bewildering-looking 10d Dirac spinor

$$\lambda_{a\hat{I}} \tag{5}$$

where a runs over the four components of a 4d Dirac spinor and $\hat{I}$ over the 8 components of a 6d Dirac spinor (thus 32 components altogether). The fact that we are dealing with a Weyl spinor means that we need not worry about many of these components; to figure out exactly which ones we should keep track of we note that under the decomposition $SO(1,9) \to SO(1,3) \times SO(6)$, the 16_+ 10d Weyl breaks down

$$16_+ \to (2_+ \otimes 4_+) \oplus (2_- \otimes 4_-) \tag{6}$$

where $2_\pm$ and $4_\pm$ respectively denote positive and negative chirality 4d and 6d spinors. Note that the total chirality of the 10d representation is always positive (chirality is "additive"), which is what we want. We now introduce yet more notation: $\alpha, \dot{\alpha}$ denote normal 4d Weyl spinor indices in keeping with standard notation, and $I\pm$ denotes positive and negative chirality 6d Weyl spinor indices. Thus I goes from 1 to 4 (whereas $\hat{I}$ includes $I+$ and $I-$ and so goes from 1 to 8). Thus before we impose any sort of Majorana condition, we have a total of 8 Weyl spinors in 4d:

$$\lambda_{I+} = \begin{pmatrix} \psi_\alpha^{I+} \\ 0 \end{pmatrix} \qquad \lambda_{I-} = \begin{pmatrix} 0 \\ \chi_{I-}^{\dot{\alpha}} \end{pmatrix} \tag{7}$$

The Majorana condition should relate these two each other. To see how that works, we write the 10d charge conjugation matrix in terms of its 4d counterpart as

$$_{(10)}C_{ab\hat{I}\hat{J}} = \ _{(4)}C_{ab} \otimes \begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix}_{\hat{I}\hat{J}} \tag{8}$$

where we have picked a basis on the 6d part so that the block diagonal structure is a division into different chiralities $I\pm$ (compare with the 4d gamma matrices in a Weyl basis)[1]. The Majorana condition reads $\lambda = C(\lambda^*)$. This relates ψ^{I+} to χ_{I-} above, letting us express both 6d chirality spinors in terms of 4 4d Weyl spinors λ_I.

$$\lambda_{I+} = \begin{pmatrix} \lambda_\alpha^I \\ 0 \end{pmatrix} \qquad \lambda_{I-} = \begin{pmatrix} 0 \\ \lambda_I^{\dot{\alpha}} \end{pmatrix} \tag{9}$$

We now decompose various 10d Γ matrices. The following decomposition satisfies the Clifford algebra

$$_{(10)}\Gamma^\mu_{ab\hat{I}\hat{J}} = \ _{(4)}\gamma^\mu_{ab} \otimes 1_{\hat{I}\hat{J}} \tag{10}$$

$$_{(10)}\Gamma^i_{ab\hat{I}\hat{J}} = \ _{(4)}\gamma^5_{ab} \otimes \ _{(6)}\Gamma^i_{\hat{I}\hat{J}} \tag{11}$$

$$\tag{12}$$

where μ, i are coordinate indices running from 0 to 3 (noncompact dimensions) and 4 to 10 (compact dimensions) respectively, and the subscripts under the Γ matrices denote their dimensionality. We are finally ready to attack the kinetic term $\bar{\lambda}\Gamma^M D_M \lambda$. Treating $M = \mu$ first, we obtain

$$\bar{\lambda}\Gamma^\mu D_\mu \lambda = (\lambda_{a\hat{I}})^\dagger \gamma^0_{ab}\gamma^\mu_{bc} D_\mu \lambda_{c\hat{I}} = \bar{\lambda}_{I\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} D_\mu \lambda^I_\alpha + \lambda^\alpha_I (\bar{\sigma}^\mu)_{\alpha\dot{\alpha}} D_\mu \bar{\lambda}^{\dot{\alpha}}_I \tag{13}$$

where in the second equality we have decomposed $\hat{I}$ into a sum over $I\pm$ and used the relation (9). This is precisely the kinetic term for four 4d Weyl spinors, with a manifest SU(4) symmetry. Next we work on the terms where $\mu = i$; these are more interesting, and we obtain

$$\bar{\lambda}\Gamma^i D_i \lambda = (\lambda_{a\hat{I}})^\dagger \gamma^0_{ab}\gamma^5_{bc}\Gamma^i_{\hat{I}\hat{J}} D_i \lambda_{c\hat{J}} = (\lambda_{aI+})^\dagger \gamma^0_{ab}\gamma^5_{bc}\Gamma^i_{I+J-} D_i \lambda_{cJ-} + (\lambda_{aI-})^\dagger \gamma^0_{ab}\gamma^5_{bc}\Gamma^i_{I-J+} D_i \lambda_{cJ+} \tag{14}$$

In the second equality we have used the fact that a single gamma matrix always relates spinors of different chirality, and thus we can decompose the 6d Γ into only "cross terms" that are Γ_{I+J-} or Γ_{I-J+}:

$$\Gamma_{\hat{I}\hat{J}} = \begin{pmatrix} 0 & \Gamma_{I-J+} \\ \Gamma_{I+J-} & 0 \end{pmatrix} \tag{15}$$

Some normal 4d gamma algebra now results in

$$-\bar{\lambda}_{I\dot{\alpha}}\Gamma^i_{I+J-} D_i \bar{\lambda}^{J\dot{\alpha}} - \lambda^\alpha_I \Gamma^i_{I-J+} D_i \lambda_{J\alpha} \tag{16}$$

This looks promising. To make it look slightly nicer, we now relate Γ^{I+J-} to Γ^{I-J+}. This is done using the form for the charge conjugation matrix (8). By using the fact that $C\Gamma C^{-1} = \text{phases} \times \Gamma^*$, it is not hard to convince yourself that in the basis we are using $\Gamma^i_{I+J-} \equiv \Gamma^i_{IJ} = (\Gamma^i_{I-J+})^*$. Thus we obtain for this part of the action

$$-i\left(\bar{\lambda}_{I\dot{\alpha}}\Gamma^i_{IJ}[X_i, \bar{\lambda}^{J\dot{\alpha}}] + \lambda^\alpha_I (\Gamma^i_{IJ})^*[X_i, \lambda^J_\alpha]\right) \tag{17}$$

Putting these pieces together one obtains the full action for the $\mathcal{N} = 4$ theory, including the fermionic terms. It may look unfamiliar because we have written things in Weyl notation (compare e.g. with (3.1) in [1]).

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[1] I believe writing such a form for the 10d charge conjugation matrix is justified given that the 6d charge conjugation matrix changes the chirality of the 6d spinor

II. $U(1)$ R SYMMETRY

The question here is: the $\mathcal{N} = 4$ SUSY algebra has a U(4) R symmetry given by the following action on the generators Q_I:

$$Q_I \to U_I^J Q_J \tag{18}$$

where U_I^J is a $U(4)$ matrix. The λ_I transform as the **4** under this U(4). The question is: is this symmetry of the SUSY algebra also respected by the action? Consider the $SO(6)$ vector X^i. This vector arose as the antisymmetric product of two **4**'s. Thus under the diagonal phase rotation $U = e^{i\phi}$, we expect something like $X \to e^{2i\phi}$. The action is obviously not invariant under such a mutilation of X^i. On the other hand, the $SU(4)$ part of this R symmetry *is* respected by the action, essentially because of careful use of $SU(4)$ gamma matrix machinery to tie together the interactions of X^i with those of λ_I (the corresponding machinery does not exist for the full $U(4)$).

III. β FUNCTION

Since the focus of this class is on the wonderful things that one can do with field theory *without* ever touching a Feynman diagram, we will not derive the beta function here but instead just look up the beta function for a gauge theory with N_f Weyl fermions and N_s complex scalars in a book. The answer is (see [2] p 106)

$$\mu \frac{dg}{d\mu} = -\frac{b}{16\pi^2} g^3 + \mathcal{O}(g^5) \tag{19}$$

where the coefficient b is given by

$$b = \frac{11}{6} T(\text{adj}) - \frac{1}{3} \sum_a T(r_a) - \frac{1}{6} \sum_n T(r_n) \tag{20}$$

a runs over fermions, n over scalars, and $T(r)$ is the index of the representation r. In our case everything is in the adjoint, and we obtain

$$b = \frac{T(\text{adj})}{6} (11 - 2N_f - N_s) \tag{21}$$

putting $N_f \to 4$, $N_s \to 3$ (three *complex* scalars, or six real ones) we obtain a vanishing 1-loop beta function.

IV. $\mathcal{N} = 4 \supset \mathcal{N} = 1$

$\mathcal{N} = 4$ has six real scalars, four Weyl fermions, and one gauge field. To match these to known $\mathcal{N} = 1$ multiplets, it looks like we need one vector multiplet V and three chiral multiplets Φ_M (for this problem M is an SU(3) index, I, J, α etc. retain their previous meanings and a is an SU(N) adjoint index); we then end up with 3 complex scalars X_M, a total of four fermions (one λ from the vector, one ψ_M from each chiral) and one gauge field. The action for such a configuration is also very restricted. If we require an SU(3) symmetry along the three chiral multiplets and no mass terms we really have very little choice for the action: when written in superfield notation (see e.g. [3] p47) it becomes

$$\mathcal{L} = \frac{1}{16} \int d^2\theta \text{tr}(W^\alpha W_\alpha) + c.c. + \int d^4\theta (\Phi^M)^\dagger e^V \Phi_M + \int d^2\theta \frac{\sqrt{2}}{3} \epsilon_{MNP} f_{abc} \Phi_M^a \Phi_N^a \Phi_P^a + c.c. \tag{22}$$

The only real freedom we had here was picking the coefficient of the cubic superfield term. For the purposes of this problem I have also set the gauge coupling g to 1.

Now: what are the symmetries of this action? There is a manifest $SU(3)$, but we expect the existence of a hidden $SU(4)$, allowing us to rotate all *four* fermions λ and ψ_M into each other (and having a nontrivial action on the X's); this is not at all apparent in the superfield formalism. To see it, we need to expand the Lagrangian out into components. It is then rather easy to see that the kinetic terms for the fermions are happy with such a large symmetry. We will thus focus on the Yukawas: since one of these comes from the Kahler term and one from the superpotential term, rotating them into each other appears to be a nontrivial business.

The relevant terms from the Lagrangian work out to be ([3], p31 and p50)

$$\mathcal{L}_Y = \sqrt{2} f_{abc} (X^{aM})^\dagger \psi^b_M \lambda^c + i\sqrt{2}\epsilon_{MNP} f_{abc} \psi^a_M \psi^b_N X^c_P + c.c. \tag{23}$$

Now, first we should note that in addition to the manifest SU(3) this also has a U(1) symmetry, under which the fields have charges

$$X_M : 2 \qquad \psi_M : -1 \qquad \lambda : 3 \tag{24}$$

On the other hand, imagine decomposing an $SU(4) \to SU(3) \times U(1)$; in that case the fundamental **4** of the SU(4) decomposes as

$$4 \to 3_{-1} \oplus 1_3 \tag{25}$$

where the main number is the representation of SU(3) and the subscript denotes the U(1) charge. This is explained using powerful machinery in [4], p439 but can be understood in a lowbrow way if you imagine simply writing an SU(4) matrix and taking the top-left corner to be the SU(3). Any diagonal $U(1)$ must have overall phase 0 in the full SU(4) matrix, which means that it must act three times as strongly (and with opposite sign) on the SU(3) singlet as on the triplet.

Note this decomposition is consistent with (λ, ψ_M) secretly forming an $SU(4)$ fundamental; the SU(3) singlet λ has (minus) three times the U(1) charge as the SU(3) triplet ψ_M. Simultaneously this tells us that X_M must me embedded into the full $SU(4)$ in a more complicated way. We will now introduce some notation and figure out what that is.

Let us compare (23) with the previously obtained (16). These look rather similar, if we enumerate the 6 X's with a notation where $X^{M,\bar{M}}$, $X^{\bar{M}} = (X^M)^\dagger$ and recall that λ is secretly the 4 component of an $SU(4)$. In that case direct comparison of the terms lets us define tentative "gamma" matrices

$$\Gamma^{\bar{M}}_{N-,4+} = -\sqrt{2}\delta^M_N \qquad \Gamma^M_{N-,P+} = i\sqrt{2}\epsilon_{MNP} \tag{26}$$

$$\Gamma^M_{N+,4-} = -\sqrt{2}\delta^M_N \qquad \Gamma^{\bar{M}}_{N+,P-} = i\sqrt{2}\epsilon_{MNP} \tag{27}$$

And indeed it is not too hard to show (remembering how these form a full 6d gamma matrix from (15)) that these purported gamma matrices really do have an algebra

$$\{\Gamma^{\bar{M}}, \Gamma^N\} = 2\delta^{MN} \qquad \{\Gamma^{\bar{M}}, \Gamma^{\bar{N}}\} = \{\Gamma^M, \Gamma^N\} = 0 \tag{28}$$

This is nothing but the fermionic oscillator algebra for three oscillators, which is equivalent up to a change of basis to the 6d Clifford algebra. Thus the random coefficients of the Yukawa terms in the action above are actually the coefficients of 6d gamma matrices, and the action is guaranteed to be invariant under an $SO(6) = SU(4)$ rotation.

V. EXTREMAL = BPS

This is a fairly involved problem. Our mission is to find solutions to the Killing spinor equation [2]

$$\left(\nabla_\rho + \frac{i}{8} F_{\hat{\mu}\hat{\nu}} \gamma^{\hat{\mu}} \gamma^{\hat{\nu}} \gamma_\rho \right) \epsilon = 0 \tag{29}$$

where the covariant deriative acting on spinors is defined to be

$$\nabla_\rho \epsilon = \left(\partial_\rho + \frac{1}{4} \omega_\mu^{\hat{\nu}\hat{\sigma}} \gamma_{\hat{\nu}\hat{\sigma}} \right) \epsilon \tag{30}$$

We will use notation where hatted indices $\hat{\mu}$ denote local Lorentz frame indices and unhatted indices μ represent coordinate indices. The solution we found last time has metric given by

$$ds^2 = -\frac{1}{H(\rho)^2} dt^2 + H(\rho)^2 \left[d\rho^2 + d\Omega_2 \right] \tag{31}$$

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[2] By hitting this equation with $(1 + \gamma^5)$, doing some gamma matrix algebra, and assuming that ϵ is a spinor of definite chirality it is possible to show that this is equivalent to the expression in the problem set, provided we introduce a factor of i in front of the $\star F$.

which means that the tetrad is given by

$$e^{\hat{t}} = \frac{1}{H}dt \qquad e^{\hat{\rho}} = H d\rho \qquad e^{\hat{\theta}} = \rho H d\theta \qquad e^{\hat{\phi}} = \rho H \sin(\theta) d\phi \tag{32}$$

The spin connection is found by demanding that $de^{\hat{a}} + \omega^{\hat{a}}_{\hat{b}} \wedge de^{\hat{b}} = 0$, and can be worked out to be

$$\omega^{\hat{t}}_{\hat{\rho}} = \omega^{\hat{\rho}}_{\hat{t}} = -\frac{\partial_\rho H}{H^3} dt \tag{33}$$

$$\omega^{\hat{\theta}}_{\hat{\rho}} = -\omega^{\hat{\rho}}_{\hat{\theta}} = \left(1 + \frac{\partial_\rho H}{H}\right) d\theta \tag{34}$$

$$\omega^{\hat{\phi}}_{\hat{\rho}} = -\omega^{\hat{\rho}}_{\hat{\phi}} = \left(1 + \frac{\partial_\rho H}{H}\right) \sin(\theta) d\phi \tag{35}$$

$$\omega^{\hat{\phi}}_{\hat{\theta}} = -\omega^{\hat{\theta}}_{\hat{\phi}} = \cos(\theta) d\phi \tag{36}$$

There is a background electric field satisfying $F_{\rho t} = -2\frac{\partial_\rho H}{H^2}$. Finally, we work in a chiral basis where the 4d gamma matrices are given by

$$\gamma^{\hat{0}} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix} \qquad \gamma^{\hat{i}} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad , \tag{37}$$

where the σ^i are the old-fashioned Pauli matrices. The metric signature is thus $(-,+,+,+)$. To go back and forth between hatted and unhatted indices we use the tetrad; furthermore we pick an orientation such that $(t, \rho, \theta, \phi) = (0, x, y, z)$. We are now ready to begin.

A. Warmup: Spinors in flat space

We begin in the gentlest possible way by solving (29) in flat space in Cartesian coordinates. This is not very difficult; the solution is

$$\epsilon = \text{const} \tag{38}$$

This simply means that Minkowski space has as many Killing spinors as the dimension of the spinor representation. We will now reproduce this result using spherical coordinates to give us some practice with covariant spinor derivatives. The relevant spin connection is given by putting $H \to 1$ in the formulas above. It is fairly easy to see that the ρ and t components of (29) are trivial: $\partial_\rho \epsilon = \partial_t \epsilon = 0$. Some algebra shows that the nontrivial components work out to be

$$\left(\partial_\theta - \frac{i\sigma_z}{2} \otimes \mathbb{I}_2\right) \epsilon = 0 \tag{39}$$

$$\left(\partial_\phi + \frac{i}{2}(\sigma_y \sin\theta - \sigma_x \cos\theta) \otimes \mathbb{I}_2\right) \epsilon = 0 \tag{40}$$

Note that the 4×4 gamma matrices have factored into tensor products of 2×2 matrices; furthermore, none of the terms in these equations mix the two chiralities of the Dirac spinor, and thus we can simply consider each of the two Weyl spinors separately. From now on we also work in the standard basis where σ_z is diagonal, and we denote the two components of the Weyl spinor in this basis by $\chi = (\chi_+, \chi_-)$. Examination of (39) then tells us that

$$\chi_+ = \exp\left(\frac{i\theta}{2}\right) f_+(\phi) \qquad \chi_- = \exp\left(-\frac{i\theta}{2}\right) f_-(\phi), \tag{41}$$

where $f_\pm$ remain to be determined. We now expand (40) to obtain

$$\left[\partial_\phi - \frac{i}{2}\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}\right] \chi = 0 \tag{42}$$

This equation has two solutions,

$$\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}(\phi+\theta)} \\ e^{\frac{i}{2}(\phi-\theta)} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{\frac{i}{2}(-\phi+\theta)} \\ -e^{\frac{i}{2}(-\phi-\theta)} \end{pmatrix} \tag{43}$$

Thus we have found two complex solutions for the two complex degrees of freedom in the Weyl spinor, precisely as expected. Note that the equations for the two Weyl spinors in the Dirac representation decoupled completely, and thus either of these two solutions can also be used for the other Weyl spinor; thus we have a total of four complex (or eight real) supersymmetries preserved by flat space.

B. Extremal Reissner-Nordstrom Black Hole

It is time to move on to the real problem. The main new ingredient here is really the electric field; denoting for convenience a matrix $F \equiv F_{\hat{\mu}\hat{\nu}}\gamma^{\mu}\gamma^{\nu}$, we plug in our solution for $F_{\rho t}$ to eventually obtain

$$F = -4\frac{\partial_\rho H}{H^2}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \tag{44}$$

Now we use this and compute the t component of (29). A slight amount of algebra gives

$$\left(\partial_t - \frac{1}{2}\frac{\partial_\rho H}{H^3}\begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}\left(1 + i\gamma^{\hat{t}}\right)\right)\epsilon = 0 \tag{45}$$

Now it seems clear that we do *not* want the Killing spinors for a static spacetime to depend on time; thus it appears that we will need the spinor ϵ to be annihilated by the messy-looking matrix. Luckily this is not difficult to arrange, as the factor $(1 + i\gamma^{\hat{t}})$ is a projector–thus any spinor ψ defined using the orthogonal projector

$$\epsilon(\rho,\theta,\phi) = \left(1 - i\gamma^{\hat{t}}\right)\psi(\rho,\theta,\phi) \tag{46}$$

will satisfy the above equation, as can be easily verified by noting that $\gamma^{\hat{t}}$ squares to -1. Here ψ is some Dirac spinor whose form we will now fix. Note that this form cuts down by a factor of two the dimension of the solution space; the projector ties together the two Weyl spinors in the Dirac spinor, and we cannot choose them separately as we did in Minkowski space. More explicitly, if we write $\epsilon = (\epsilon_1, \epsilon_2)$, then the constraint (46) simply means that $\epsilon_1 = -i\epsilon_2$.

We now move on to the radial equation; this will allow us to fix the radial profile of the solution. Some uninteresting algebra reveals eventually

$$\left[\partial_\rho - \frac{i}{2}\frac{\partial_\rho H}{H}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\epsilon = 0 \tag{47}$$

Note that this equation has two solutions

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{H(\rho)} \\ i/\sqrt{H(\rho)} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sqrt{H(\rho)} \\ -i\sqrt{H(\rho)} \end{pmatrix} \tag{48}$$

but of these only the first one is consistent with the projection (46). Soothingly both of these solutions become constant at infinity. We move on now to the angular equations; these become

$$\left[\left(\partial_\theta - \frac{i\sigma_z}{2}\otimes\mathbb{I}_2\right) - i\frac{\rho\partial_\rho H}{H}\frac{\sigma_z\otimes\mathbb{I}_2}{2}\left(1 + i\gamma^{\hat{t}}\right)\right]\epsilon = 0 \tag{49}$$

$$\left[\left(\partial_\phi + \frac{i}{2}(\sigma_y\sin\theta - \sigma_x\cos\theta)\otimes\mathbb{I}_2\right) + i\sin(\theta)\frac{\rho\partial_\rho H}{H}\frac{\sigma_y\otimes\mathbb{I}_2}{2}\left(1 + i\gamma^{\hat{t}}\right)\right]\epsilon = 0 \tag{50}$$

These both have a very nice structure; the second half of each equation is automatically satisfied by virtue of the projection (46), and the first half is precisely the flat-space angular equation that we have already solved. Thus we can immediately take over our results from the previous section. This fixes the angular dependence of $\epsilon_{1,2}(\theta,\phi)$; thus we now have a complete solution.

The critical difference between this and the Minkowski space example is that we had to project out half of the degrees of freedom to obtain a solution. We could not fix both Weyl spinors in the Dirac spinor independently; having specified one, the projection determines the other. Thus this background is invariant only under half as many supersymmetries as flat space, and is annihilated by half of the SUSY generators. This is precisely what one finds for a BPS state; we conclude the that the extremal Reissner-Nordstrom black hole is BPS (Hooray!).

C. $AdS_2 \times S^2$, or The Return of Supersymmetry

We now squint closely at the "near-horizon" region of the geometry, i.e. $\rho \to 0$. For the first time we now need the explicit form of H, which is $H(\rho) = 1 + \frac{A}{\rho}$. Putting this into (45) we find something miraculous–in the $\rho \to 0$ limit, the coefficient of the projector vanishes! Thus we no longer need to project away half of the modes. We can now use both of the radial solutions that we found earlier in (48). Note this gives us two different solutions, which satisfy respectively $\epsilon_2 = \pm i\epsilon_1$ (only the first of these two solutions was consistent with the projection).

Plugging in these two forms and taking the near-horizon limit in the angular equations leaves us with

$$\left(\partial_\theta \mp i\frac{\sigma_z}{2}\right)\epsilon_1 = 0 \tag{51}$$

$$\left(\partial_\phi - \frac{i}{2}\begin{pmatrix} 0 & e^{\pm i\theta} \\ e^{\mp i\theta} & 0 \end{pmatrix}\right)\epsilon_1 = 0 \tag{52}$$

where one should consistently take the top or the bottom of all the $\pm$ depending on which of the two radial solutions is being considered. These equations admit the following solutions

$$\epsilon_1 = \begin{pmatrix} e^{\frac{i}{2}(\phi\pm\theta)} \\ e^{\frac{i}{2}(\phi\mp\theta)} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{\frac{i}{2}(-\phi\pm\theta)} \\ -e^{\frac{i}{2}(-\phi\mp\theta)} \end{pmatrix} \tag{53}$$

We now have a total of $2 \times 2 = 4$ (picking either of $\pm$ and one of the two solutions above) complex solutions, so 8 real supersymmetries are preserved (Hooray!).

VI. W-BOSONS FROM ADJOINT HIGGSING

For the "W-bosons", the relevant term from the $\mathcal{N} = 4$ Lagrangian is

$$\mathcal{L}_{\text{interesting}} = -\frac{1}{2g_{\text{YM}}^2}D_\mu X^i D^\mu X^i = -\frac{1}{2g_{\text{YM}}^2}\text{tr}([A_\mu, X^i][A^\mu, X^i]) + \text{boring stuff} \tag{54}$$

Now imagine that $X^i = \text{diag}(x_1^i \ldots x_N^i)$. Note that unlike the "normal" Higgs mechanism we have some freedom in the choice of X; if it is diagonal then the potential term in the action vanishes for any choice of x^i. Consider also A to be an $N \times N$ matrix $A = A_{ab}$. In that case a quick multiplication shows that

$$[A, X^i]_{ab} = A_{ab}(x_b^i - x_a^i) \tag{55}$$

and thus we get a term

$$\mathcal{L}_{\text{interesting}} = \frac{1}{2g_{YM}^2}\sum_{i;a,b} A_{ab}A_{ba}(x_a^i - x_b^i)^2 \tag{56}$$

Now let's consider the gauge group to be SU(N) for concreteness. In that case $A_{ab} = A_{ba}*$, and this is precisely a mass term for the ab gauge bosons, as claimed (where we are considering each off-diagonal element ab of the matrix and its transpose ba together to be a single complex degree of freedom). Next we consider the scalar masses; here the relevant term is

$$\mathcal{L}_{\text{interesting}} = \frac{1}{4g_{\text{YM}}^2}\sum_{i\neq j}\text{tr}([X^i, X^j]^2) \tag{57}$$

Now we expand this out to quadratic order in perturbations δX^i around the background values X^i. Essentially there are two different types of terms, which work out after some index shuffling to be

$$\mathcal{L}_{\text{interesting}} = \frac{1}{4g_{\text{YM}}^2}\sum_{i\neq j}\text{tr}\left([X^i, \delta X^j]^2 + [X^i, \delta X^j][\delta X^i, X^j]\right) \tag{58}$$

Now we work out these commutators; for notational convenience we define $\Delta_{ab}^i \equiv x_a^i - x_b^i$ and expand, finding eventually that

$$\mathcal{L}_{\text{interesting}} = \frac{1}{4g_{\text{YM}}^2}\sum_{i,j;a,b}\delta X_{ab}^i(\delta X_{ab}^j)^*\left[\Delta^i\Delta^j - \delta^{ij}\vec{\Delta}^2\right]_{ab} \equiv -\frac{1}{4g_{\text{YM}}^2}\sum_{i,j;a,b}\delta X_{ab}^i(\delta X_{ab}^j)^* m_{ab}^{ij} \tag{59}$$

Thus for fixed a, b we obtain a mass matrix; to understand more clearly the structure of this matrix it is easiest to imagine doing an $SO(6)$ rotation so that $\Delta^i = (\Delta, 0, 0, ..., 0)$. In that case $m^{ij} = \mathrm{diag}(0, \Delta^2, \Delta^2, ...\Delta^2)$; thus we obtain for each ab 5 scalars with the same mass as the gauge boson and one massless scalar.

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