PROBLEM SET 10

Not due: This problem set will **not** be collected or graded.

Reading: You may find Chapter 12 of Carroll & Ostlie (*Introduction to Modern Astrophysics*) helpful for background reading on star formation.

Reminders: The final exam will be on Wednesday, May 24, 9:00am-12:00pm.

1. **Mass loss rate in Roche lobe overflow.** Consider a stellar binary comprising stars of mass M_1 and M_2 located at r_1 and r_2, respectively, where we use a reference frame rotating with the orbital angular frequency Ω whose origin is at the binary center of mass. In this coordinate system, the orbit lies in the x-y plane and the stars are at fixed positions on the x-axis. The Roche potential is

$$\Phi(r) = -\frac{GM_1}{|r - r_1|} - \frac{GM_2}{|r - r_2|} - \frac{1}{2}|\Omega \times r|^2,$$

where we have neglected the Coriolis force. Star 2 fills its Roche lobe and loses matter through the inner Lagrangian (L_1) point, located at $(x_L, 0, 0)$ in Cartesian coordinates.

   (a) Recall that the L_1 point has $\nabla\Phi = 0$. Show that, to order of magnitude, the potential at a nearby point $(x_L, \Delta y, 0)$ is

   $$\Phi(x_L, \Delta y, 0) \simeq \Omega^2 (\Delta y)^2$$

   (b) Use an energy argument to show that mass escapes from star 2 from a patch near the L_1 point with radius

   $$\Delta y \sim c_s/\Omega,$$

   where c_s is the sound speed in the patch.

   (c) Show that, to order of magnitude, the mass loss rate from star 2 through the L_1 point is

   $$-\dot{M_2} \sim \frac{\rho c_s^3 P_{\text{orb}}^2}{4\pi},$$

   where P_{orb} is the binary period and ρ is the mass density in the patch. The sound speed is order the mean thermal velocity and thus scales with temperature as $T^{1/2}$. Since both ρ and T increase below the stellar surface, the mass loss rate will grow if the Roche lobe penetrates deep into the "natural size" of the star.

2. **The thin disk approximation for accretion disks.**

   Consider an accretion disk around a star of mass M, with the disk's midplane having $z = 0$ in a rectangular coordinate system with origin at the star. At a midplane radius r, the disk has height $H(r)$ in the z-direction (so that its thickness is $2H$. We will assume that $H \ll r$, and that the mass in the disk is negligible compared to M (so that the disk's self-gravity may be neglected).

   (a) Write down an equation for hydrostatic equilibrium in the z direction in terms of the pressure P, density ρ, and mid-plane radius r.

(b) Show that the disk height may be written as

$$H(r) \simeq \left(\frac{c_s}{v_K}\right) R,$$

where $c_s \simeq \sqrt{P/\rho}$ is the sound speed and v_K is the Kepler velocity.

(c) Show that the thin disk condition ($H \ll r$) is equivalent to requiring that the Keplerian orbital velocity is highly supersonic ($v_K \gg c_s$).

3. **Star formation: the collapse of a molecular cloud.**

Recall that the radial hydrodynamic equation of motion is

$$\rho \ddot{r} = -\frac{GM_r \rho}{r^2} - \frac{dP}{dr},$$

where ρ is the density, P is the pressure, and M_r is the mass interior to radius r (assuming spherical symmetry). For the case of hydrostatic equilibrium, $\ddot{r} = 0$. (Note that for a collapsing cloud, the mass enclosed by the collapsing surface remains constant as the density increases.)

Consider the collapse of a uniform-density giant molecular cloud with temperature $T = 150$ K and number density 10^8 cm^{-3}.

(a) Using the ideal gas law, estimate $|dP/dr| \approx |\Delta P/\Delta r| \sim P_c/R_J$ at the start of the cloud's collapse, where P_c is an approximate value for the central pressure of the cloud and R_J is the Jeans length. Assume that $P = 0$ at the edge of the cloud, and use the Jeans values for its mass and radius.

(b) Show that, within the accuracy of our crude estimates for the Jeans parameters, the value of $|dP/dr|$ you found above is comparable to (i.e. within an order of magnitude of) what is required for quasi-hydrostatic equilibrium.

(c) As long as the collapse remains isothermal, show that the size of dP/dr continues to decrease relative to gravitational effects, so that the pressure gradient may be neglected once the free-fall collapse begins.

4. **Magnetic fields and star formation.**

Estimate the gravitational potential energy per unit volume for the pre-collapse giant molecular cloud described in the previous problem. (Take M to be the Jeans mass for this cloud.) Compare this to the magnetic energy density of the cloud if it is threaded by a uniform magnetic field of strength $B = 10$ μG. Would you expect that the magnetic field could play a siginificant role in the collapse of this cloud?

5. **Angular momentum and star formation.**

In this problem, we will explore the effect of rotation on the collapse of a molecular cloud.

(a) Write down the radial equation of motion for free-fall of a molecular cloud, including a centrifugal term for a non-zero fixed-axis angular velocity of the cloud. Using conservation of angular momentum, show that the collapse of the cloud will stop in the plane perpendicular to the axis of rotation when the cloud radius reaches

$$r_f = \frac{\omega_0^2 r_0^4}{2GM_r},$$

where ω_0 and r_0 are the original angular velocity and radius of the cloud. You may assume that the initial radial velocity of the cloud is zero, and that $r_f \ll r_0$. You may also assume (incorrectly) that the cloud rotates as a rigid body throughout the collapse.

Hint: You may use the fact that

$$\frac{d^2r}{dt^2} = \frac{dr}{dt} \times \frac{d}{dr}\left(\frac{dr}{dt}\right) = v_r \frac{dv_r}{dr},$$

where v_r is the radial velocity.

(b) Assume that the original cloud had a mass of $1 M_\odot$ and a radius of 0.5 pc. If the collapse is halted at a radius of approximately 100 AU, find the initial angular velocity of the cloud. What was the original rotational velocity (in cm s^{-1}) of the edge of the cloud?

(c) Assuming that the moment of inertia is that of a uniform sphere when the collapse begins, and that the moment inertia is that of a uniform disk when the collapse ends, determine the rotational velocity at 100 AU when the collapse stops.

(d) After the collapse has stopped, calculate the time required for a mass element at 100 AU to make one complete revolution around the center of the protostar. Compare this time to the orbital period at 100 AU expected from Kepler's third law. Why would you not expect the two periods to be identical?

6. **Deducing the nature of a highly reddened star.**

Note: To do this problem, you will need to look up tabulated information on the typical intrinsic colors and absolute magnitudes of stars as a function of spectral type and luminosity class. There are any number of references where you can find this information. One particularly convenient source is the fourth edition of *Allen's Astrophysical Quantities* (2000; ed. Arthur N. Cox, AIP Press and Springer-Verlag), Tables 7.6, 7.7, 7.8, and 15.7.

Infrared observations are made of a star lying toward the Galactic center, yielding the following infrared (apparent) magnitudes: $J = 19.21\pm0.05$, $H = 16.16\pm0.05$, and $K = 14.69\pm0.05$. In this problem, you will use the interstellar reddening law to compare the observed magnitudes to the tabulated intrinsic (unreddened) colors and absolute magnitude of stars to determine the most likely range of spectral type and luminosity class for this star. Recall from our in-class discussion of the reddening law that in the infrared: $A_J = 0.282\, A_V$, $A_H = 0.175\, A_V$, and $A_K = 0.112\, A_V$.

(a) Recall that the intrinsic (unreddened) apparent magnitude $m_{\lambda,0}$ in band λ is related to the observed magnitude m_λ by $m_\lambda = m_{\lambda,0} + A_\lambda$, where A_λ is the reddening. Using this relation as well as the interstellar extinction law and the measured $J - H$ and $H - K$ colors for the star, derive a linear equation relating the *intrinsic* colors of the star, $(J - H)_0$ and $(H - K)_0$. Be sure to propagate the measurement uncertainties into this equation as well. This equation represents the allowed locus of intrinsic colors for our star, as a function of the reddening A_V. Show this allowed region as a band on a color-color diagram that plots $(J - H)$ vs $(H - K)$ for a range corresponding to A_V from 15 to 25.

(b) What could this star be? As candidates, consider main sequence stars (luminosity class V), giants (luminosity class III), and supergiants (luminosity class I). For each of these classes, compare our star's allowed region of intrinsic $J - H$ and $H - K$ colors to the tabulated intrisic infrared colors

of the various spectral types by placing both on the same color-color plot. For each luminosity class, what range of spectral types has intrinsic colors consistent with the observed colors of the star, and for what values of the reddening A_V?

(c) For each luminosity class, determine the distance to the star (in kpc) for each allowed spectral type. Do this by comparing the observed K magnitude of the star with the absolute K magnitude (M_K) for each spectral type/luminosity class. (It is generally the absolute V magnitude that is tabulated; you will need to combine this with the intrinsic $V - K$ color and the reddening in K.) You should be able to argue that some of these distances are implausible given the amount of reddening and/or the direction in the sky. Use such arguments to infer what the most plausible range of spectral class and luminosity class for the star.