

The first part of this course will cover the foundational material of homogeneous big bang cosmology. There are three basic topics:

1. General Relativity
2. Cosmological Models with Idealized Matter
3. Cosmological Models with Understood Matter

## 1 General Relativity

References:

- Landau and Lifshitz, Volume 2: *The Classical Theory of Fields*
- Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*
- Misner, Thorne, and Wheeler, *Gravitation*

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This will be a terse introduction to general relativity. It will be logically complete, and adequate for our later purposes, but a lot of good stuff is left out (astrophysical applications, tests, black holes, gravitational radiation, ...).

### 1.1 Transformations and Metrics

We want equations that are independent of coordinates. More precisely, we want them to be invariant under “smooth” reparameterizations

$$x'^{\mu} = x'^{\mu}(x)$$

To do local physics we need derivatives. Of course

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \quad (\text{Note: summation convention})$$

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} \equiv R^{\mu}_{\nu} \in GL(4) \quad (\text{invertible real matrix})$$

We want to have special relativity in small empty regions, so we must introduce more structure. The right thing to do is to introduce a symmetric, non-singular (signature + - - -), tensor field  $g_{\mu\nu}(x)$  so that we can define intervals

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (\text{“Pythagoras”})$$

For this to be invariant ( $g'_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} dx^\mu dx^\nu$ ), we need

$$g'_{\mu\nu}(x') = (R^{-1})^\alpha{}_\mu (R^{-1})^\beta{}_\nu g_{\alpha\beta}(x)$$

Since  $R^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ ,  $(R^{-1})^\rho{}_\sigma = \frac{\partial x^\rho}{\partial x'^\sigma}$  (chain rule:  $\delta^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x'^\beta} = \frac{\partial x^\lambda}{\partial x'^\beta} \frac{\partial x'^\alpha}{\partial x^\lambda}$ )

We can write the transformation law for  $g_{\mu\nu}$  in matrix form:

$$G' = R^{-1} G (R^{-1})^T$$

From linear algebra, we can insure  $G'$  is diagonal with  $\pm 1$  (or 0) entries. The signature, e.g.  $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ , is determined.

There are residual transformations that leave this form of  $g_{\mu\nu}$  intact. They are the Lorentz transformations!

Generalizing  $g_{\mu\nu}, dx^\mu$ , we define tensors of more general kinds

$$T_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n}(x)$$

by the transformation law

$$T'_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n}(x') = (R^{-1})^{\alpha_1}{}_{\mu_1} \dots (R^{-1})^{\alpha_m}{}_{\mu_m} R^{\nu_1}{}_{\beta_1} \dots R^{\nu_n}{}_{\beta_n} \cdot T_{\alpha_1 \dots \alpha_m}^{\beta_1 \dots \beta_n}(x)$$

Example: Inverse metric  $g^{\mu\nu}$

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$$

Operations:

- Multiplication by number
- Tensor, of the same type can be added
- Outer Product

$$V^{\nu_1} W^{\nu_2}{}_\mu \equiv T^{\nu_1 \nu_2}{}_\mu$$

- Contraction

$$T_{\mu_1 \mu_2 \dots \mu_m}^{\mu_1 \nu_2 \dots \nu_n} = \tilde{T}_{\mu_2 \dots \mu_m}^{\nu_2 \dots \nu_n}$$

Example:

$$g_{\mu\nu} dx^\alpha dx^\beta = T_{\mu\nu}^{\alpha\beta} \text{ a tensor}$$

$$\text{Contraction: } T_{\mu\nu}^{\mu\nu} = ds^2$$

“0” is a tensor (all components = 0).  $\delta^\mu_\nu$  is a tensor.

## 1.2 Covariant Derivatives: Affine Structure

- Scalar Field  $\phi'(x') = \phi(x)$
- Vector  $A'_\mu(x') = \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha(x) \quad ((R^{-1})^\alpha{}_\mu A_\alpha)$
- Operator  $\partial'_\nu \equiv \frac{\partial}{\partial x'^\nu} = \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial}{\partial x^\alpha}$

Is there an invariant derivative?

$$\begin{aligned} \partial'_\nu A'_\mu &= \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial}{\partial x^\alpha} \left( \frac{\partial x^\beta}{\partial x'^\mu} A_\beta \right) \\ &= \underbrace{\frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} \partial_\alpha A_\beta}_{\text{good}} + \underbrace{\frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} A_\beta}_{\text{bad (hard to use - not a tensor)}} \end{aligned}$$

Add correction term:  $\nabla_\nu A_\mu \equiv \partial_\nu A_\mu - \Gamma_{\nu\mu}^\lambda A_\lambda$ :

$$\begin{aligned} \nabla'_\nu A'_\mu &= S^\alpha{}_\nu S^\beta{}_\mu \partial_\alpha A_\beta + \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu} A_\sigma - \Gamma'^{\lambda}{}_{\nu\mu} S^\sigma{}_\lambda A_\sigma \\ &\stackrel{?}{=} S^\alpha{}_\nu S^\beta{}_\mu (\partial_\alpha A_\beta - \Gamma_{\alpha\beta}^\sigma A_\sigma) \end{aligned}$$

where  $S^\alpha{}_\nu \equiv (R^{-1})^\alpha{}_\nu = \frac{\partial x^\alpha}{\partial x'^\nu}$ .

This will work if

$$\begin{aligned} \Gamma'^{\lambda}{}_{\nu\mu} S^\sigma{}_\lambda &= S^\alpha{}_\nu S^\beta{}_\mu \Gamma_{\alpha\beta}^\sigma + \frac{\partial^2 x^\sigma}{\partial x'^\nu \partial x'^\mu} \\ \text{that is, } \Gamma'^{\lambda}{}_{\nu\mu} &= R^\lambda{}_\sigma S^\alpha{}_\nu S^\beta{}_\mu \Gamma_{\alpha\beta}^\sigma + \frac{\partial x'^\lambda}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\nu \partial x'^\mu} \end{aligned}$$

Note that the inhomogeneous part is symmetric in  $\mu \leftrightarrow \nu$ . So we can assume  $\Gamma_{\alpha\beta}^\lambda = \Gamma_{\beta\alpha}^\lambda$  consistently. (The antisymmetric “torsion” part is a tensor on its own!)

Given  $\Gamma$ , we can take covariant derivatives as

$$\nabla_\alpha T_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n} = \partial_\alpha T_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n} - \Gamma_{\alpha\mu_1}^\lambda T_{\lambda\mu_2 \dots \mu_m}^{\nu_1 \dots \nu_n} - \dots - \Gamma_{\alpha\mu_m}^\lambda T_{\mu_1 \dots \mu_{m-1} \lambda}^{\nu_1 \dots \nu_n} + \Gamma_{\alpha\lambda}^{\nu_1} T_{\mu_1 \mu_2 \dots \mu_m}^{\lambda\nu_2 \dots \nu_n}$$

This gives a tensor. We use the Leibniz rule in products.

## 1.3 Covariant Derivatives: Metric

Big result: given a metric, there is a unique preferred connection  $\Gamma$

Demand  $\nabla_\lambda g_{\mu\nu} = 0$  (and symmetry)

$$\begin{aligned} 0 &= \partial_\lambda g_{\mu\nu} - \Gamma_{\lambda\mu}^\alpha g_{\alpha\nu} - \Gamma_{\lambda\nu}^\alpha g_{\alpha\mu} \\ 0 &= \partial_\mu g_{\nu\lambda} - \Gamma_{\lambda\mu}^\alpha g_{\alpha\nu} - \Gamma_{\mu\nu}^\alpha g_{\alpha\lambda} \\ 0 &= \partial_\nu g_{\lambda\mu} - \Gamma_{\lambda\nu}^\alpha g_{\alpha\mu} - \Gamma_{\mu\nu}^\alpha g_{\alpha\lambda} \end{aligned}$$

Subtract the first line from the sum of the second and third:

$$2\Gamma_{\mu\nu}^{\alpha}g_{\alpha\lambda} = \partial_{\mu}g_{\lambda\nu} + \partial_{\nu}g_{\lambda\mu} - \partial_{\lambda}g_{\mu\nu}$$

$$\times \frac{g^{\lambda\beta}}{2} : \quad \Gamma_{\mu\nu}^{\beta} = \frac{1}{2}g^{\lambda\beta} (\partial_{\mu}g_{\lambda\nu} + \partial_{\nu}g_{\lambda\mu} - \partial_{\lambda}g_{\mu\nu})$$

Conversely, this works!

Extra term in  $\Gamma'$ :

$$\frac{1}{2}g^{\lambda\beta} \left[ \cancel{\left( \partial'_{\mu} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} \right) g_{\sigma\nu}} + \boxed{\left( \partial'_{\mu} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right) g_{\lambda\sigma}} \right. \\ \left. + \cancel{\left( \partial'_{\nu} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} \right) g_{\sigma\mu}} + \boxed{\left( \partial'_{\nu} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \right) g_{\lambda\sigma}} - \cancel{\left( \partial'_{\lambda} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \right) g_{\sigma\nu}} - \cancel{\left( \partial'_{\lambda} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right) g_{\sigma\mu}} \right]$$

The boxed terms give the desired inhomogeneous terms; the others cancel.

## 1.4 Invariant Measure

$$d^4x' = \underbrace{\det \left( \frac{\partial x'^{\mu}}{\partial x^{\nu}} \right)}_{\text{Jacobian; } = \det R} d^4x$$

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$

$$G' = R^{-1}G(R^{-1})^T$$

$$\det g'_{\mu\nu} = \frac{1}{(\det R)^2} \det g_{\mu\nu}$$

Write  $g \equiv \det(g_{\mu\nu})$ ; then

$$\sqrt{g'}d^4x' = \sqrt{g}d^4x$$

is an invariant measure.

## 1.5 Curvature

In order to make the  $g_{\mu\nu}$  into dynamical variables, we want them to appear with derivatives in the Lagrangian. Can we form tensors such that this occurs?

$$\text{Trick: } (\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})A_{\beta} \equiv - \underbrace{R^{\alpha}_{\beta\mu\nu}}_{\text{involves } g} \cdot \underbrace{A_{\alpha}}_{\text{no derivatives}}$$

This  $R^{\alpha}_{\beta\mu\nu}$  automatically transforms as a proper tensor; the “hard” part is to show that the derivatives on  $A_{\alpha}$  all cancel so we get this form.

$$\nabla_{\mu}A_{\beta} = \partial_{\mu}A_{\beta} - \Gamma_{\nu\beta}^{\lambda}A_{\lambda}$$

$$\nabla_{\mu}(\nabla_{\nu}A_{\beta}) = \partial_{\mu}(\nabla_{\nu}A_{\beta}) - \underbrace{\Gamma_{\mu\nu}^{\sigma}\nabla_{\sigma}A_{\beta}}_{\text{symmetric} \Rightarrow \text{drop it}} - \Gamma_{\mu\beta}^{\sigma}\nabla_{\nu}A_{\sigma}$$

$$= \cancel{\partial_{\mu}\partial_{\nu}A_{\beta}} - \partial_{\mu}(\Gamma_{\nu\lambda}^{\sigma}A_{\sigma}) - \Gamma_{\mu\nu}^{\sigma}\partial_{\nu}A_{\sigma} + \Gamma_{\mu\beta}^{\rho}\Gamma_{\nu\rho}^{\sigma}A_{\sigma}$$

$$- \partial_{\mu}\Gamma_{\nu\lambda}^{\sigma}A_{\sigma} - \Gamma_{\nu\lambda}^{\sigma}\partial_{\mu}A_{\sigma} - \Gamma_{\mu\lambda}^{\sigma}\partial_{\nu}A_{\sigma} + \Gamma_{\mu\beta}^{\rho}\Gamma_{\nu\rho}^{\sigma}A_{\sigma}$$

$$\text{so } R^{\alpha}_{\beta\mu\nu} = \partial_{\mu}\Gamma_{\nu\beta}^{\alpha} - \partial_{\nu}\Gamma_{\mu\beta}^{\alpha} + \Gamma_{\mu\rho}^{\alpha}\Gamma_{\nu\beta}^{\rho} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\mu\beta}^{\rho}$$

### Symmetry properties of $R_{\alpha\beta\gamma\delta}$

We can go to a frame where  $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = 0$  (at point of interest). These are “geodesic coordinates”. There we know that  $\Gamma_{\beta\gamma}^\alpha = 0$  (but not its derivatives!)

Then

$$\begin{aligned} R^\alpha{}_{\beta\mu\nu} &= \frac{1}{2} [\partial_\mu (g^{\alpha\sigma} (\partial_\nu g_{\sigma\beta} - \partial_\beta g_{\nu\sigma} - \partial_\sigma g_{\beta\nu})) - (\mu \leftrightarrow \nu)] \\ &= \frac{1}{2} g^{\alpha\sigma} [\partial_\mu \partial_\beta g_{\nu\sigma} - \partial_\mu \partial_\sigma g_{\beta\nu} - \partial_\nu \partial_\beta g_{\mu\sigma} - \partial_\nu \partial_\sigma g_{\beta\mu}] \\ &= g^{\alpha\sigma} R_{\sigma\beta\gamma\delta} \end{aligned}$$

$$\text{with } R_{\alpha\beta\gamma\delta} = \frac{1}{2} [\partial_\alpha \partial_\delta g_{\beta\gamma} + \partial_\beta \partial_\gamma g_{\alpha\delta} - \partial_\alpha \partial_\gamma g_{\beta\delta} - \partial_\beta \partial_\delta g_{\alpha\gamma}]$$

$$\Rightarrow R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$$

(e.g. Look at the coefficient of  $\partial_\alpha \partial_\delta g_{\beta\gamma}$ :

$$+1 \text{ in } R_{\alpha\beta\gamma\delta}$$

$$-1 \text{ in } R_{\alpha\gamma\delta\beta}$$

$$0 \text{ in } R_{\alpha\delta\beta\gamma}$$

Since these are tensor identities, they hold in any frame!

Also notable is the Bianchi identity;

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta} = 0$$

It follows from:

$$\begin{aligned} &[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + [\nabla_\beta, [\nabla_\gamma, \nabla_\alpha]] + [\nabla_\gamma, [\nabla_\alpha, \nabla_\beta]] = \\ &\nabla_\alpha \nabla_\beta \nabla_\gamma - \nabla_\alpha \nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma \nabla_\alpha + \nabla_\gamma \nabla_\beta \nabla_\alpha + \\ &\nabla_\beta \nabla_\gamma \nabla_\alpha - \nabla_\beta \nabla_\alpha \nabla_\gamma - \nabla_\gamma \nabla_\alpha \nabla_\beta + \nabla_\alpha \nabla_\gamma \nabla_\beta + \\ &\nabla_\gamma \nabla_\alpha \nabla_\beta - \nabla_\gamma \nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta \nabla_\gamma + \nabla_\beta \nabla_\alpha \nabla_\gamma = 0 \end{aligned}$$

e.g.:

$$\begin{aligned} \nabla_\alpha [\nabla_\beta, \nabla_\gamma] A_\mu &= -\nabla_\alpha (R^\nu{}_{\mu\beta\gamma} A_\nu) = -\nabla_\alpha \overset{(1)}{R^\nu{}_{\mu\beta\gamma}} A_\nu - R^\nu{}_{\mu\beta\gamma} \nabla_\alpha A_\nu \\ -[\nabla_\beta, \nabla_\gamma] \nabla_\alpha A_\mu &= \overset{(3)}{R^\nu{}_{\alpha\beta\gamma}} \nabla_\nu A_\mu + \overset{(4)}{R^\nu{}_{\mu\beta\gamma}} \nabla_\alpha A_\nu \end{aligned}$$

(2) cancels against (4), (3) will go away by the symmetry of  $R^\nu{}_{\alpha\beta\gamma} + R^\nu{}_{\beta\gamma\alpha} + R^\nu{}_{\gamma\alpha\beta} = 0$ , so (1) generates the Bianchi identity. This identity is the gravity analogue of

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

in electromagnetism, or  $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$  (existence of vector potential).

## 1.6 Invariant Actions

Since we have an invariant measure  $\int \sqrt{g} d^4x$  ( $\mathcal{L}$ ), we get invariant field theories by putting invariant expressions inside ( $\mathcal{L}$ ).

Given a special-relativistic invariant theory, we just need to change

$$\begin{aligned} d^4x &\rightarrow \sqrt{g} d^4x \\ \eta_{\mu\nu} &\rightarrow g_{\mu\nu} \\ \partial_\mu &\rightarrow \nabla_\mu \end{aligned}$$

to make a general-relativistic invariant theory. This is the “minimal coupling” procedure.

Examples:

1. Scalar field:  $\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$

2. Transverse vector field

$$\begin{aligned} F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma_{\mu\nu}^\sigma A_\sigma - \partial_\nu A_\mu + \Gamma_{\nu\mu}^\sigma A_\sigma \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &\text{(no need for } \Gamma, \text{ or } \partial g) \\ \mathcal{L} &= -\frac{1}{4} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \end{aligned}$$

supports gauge symmetry  $A_\mu \rightarrow A_\mu + \partial_\mu \chi$

3. Longitudinal vector field (instructive)

$$\begin{aligned} \nabla_\mu A^\mu &= \partial_\mu A^\mu + \Gamma_{\mu\alpha}^\mu A^\alpha \\ \Gamma_{\mu\alpha}^\mu &= \frac{1}{2} g^{\mu\sigma} (\partial_\alpha g_{\sigma\mu} + \cancel{\partial_\mu g_{\alpha\sigma}} - \cancel{\partial_\sigma g_{\mu\alpha}}) \\ &= \frac{1}{2} g^{\mu\sigma} \partial_\alpha g_{\sigma\mu} \\ &= \frac{1}{\sqrt{g}} \partial_\alpha \sqrt{g} \end{aligned}$$

[To prove  $\partial_\alpha g = g g^{\mu\sigma} \partial_\alpha g_{\mu\sigma}$ , use expansion by minors and expansion for inverse matrix. Check on diagonal matrices!]

So  $\nabla_\mu A^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} A^\mu)$ . Thus  $\int d^4x \sqrt{g} \nabla_\mu A^\mu = \int d^4x \partial_\mu (\sqrt{g} A^\mu)$  is semi-trivial: it is a boundary term.

$$\int d^4x \nabla_\mu A^\mu \nabla_\nu A^\nu \text{ gives dynamics.}$$

This supports a gauge transformation

$$\begin{aligned} \sqrt{g} A^\mu &\rightarrow \sqrt{g} A^\mu + \epsilon^{\mu\nu\rho\sigma} \partial_\nu \Lambda_{\rho\sigma} \\ \Lambda_{\rho\sigma} &= -\Lambda_{\sigma\rho} \end{aligned}$$

4. Gravity itself

$R = g^{\beta\gamma} R^\alpha_{\beta\alpha\gamma}$  and 1 are invariant. The latter is non-trivial, due to the measure factor  $\int d^4x \sqrt{g}$  “Cosmological term”

5. Spinors! These are foundational, but relegated to the Appendix.

## 1.7 Field Equations

### 1. Scalar field

$$S = \int d^4x \overbrace{\sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right)}^\Lambda$$

$$\partial_\mu \frac{\delta \Lambda}{\delta \partial_\mu \phi} = \frac{\delta \Lambda}{\delta \phi}$$

$$\partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) = -\sqrt{g} V'(\phi)$$

$$\text{or } \nabla_\mu (g^{\mu\nu} \partial_\nu \phi) = -V'(\phi)$$

### 2. Transverse vector field

$$S = -\frac{1}{4} \int d^4x \sqrt{g} g^{\alpha\gamma} g^{\beta\delta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\partial_\gamma A_\delta - \partial_\delta A_\gamma) - \underbrace{\int d^4x \sqrt{g} j^\mu A_\mu}_{\text{coupling to current}}$$

$$\begin{aligned} \partial_\mu \frac{\delta \sqrt{g} \mathcal{L}}{\delta \partial_\mu A_\nu} &= -\partial_\mu \left( \sqrt{g} g^{\mu\gamma} g^{\nu\delta} (\partial_\gamma A_\delta - \partial_\delta A_\gamma) \right) \\ &= -\partial_\mu (\sqrt{g} F^{\mu\nu}) \stackrel{\text{exercise!}}{=} -\sqrt{g} \nabla_\mu F^{\mu\nu} \\ \frac{\delta \sqrt{g} \mathcal{L}}{\delta \partial_\mu A_\nu} &= -\sqrt{g} j^\nu \end{aligned}$$

$$\text{Equation of motion: } \nabla_\mu F^{\mu\nu} = j^\nu$$

$$\text{or } \partial_\mu (\sqrt{g} g^{\mu\gamma} g^{\nu\delta} F_{\gamma\delta}) = \sqrt{g} j^\nu$$

As a consistency condition we have:  $\partial_\nu (\sqrt{g} j^\nu) = 0 \Rightarrow \sqrt{g} \nabla_\nu j^\nu$

### 3. Longitudinal vector field

$$\begin{aligned} \int \sqrt{g} \nabla_\mu A^\mu \nabla_\nu A^\nu &= \int \sqrt{g} \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} A^\mu) \frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} A^\nu) \\ &\equiv \int \frac{1}{\sqrt{g}} \partial_\mu p^\mu \partial_\nu p^\nu \\ \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu p^\nu} &= 2\partial_\mu \left( \frac{1}{\sqrt{g}} \delta_\nu^\mu \partial_\sigma p^\sigma \right) \\ &= 2\partial_\nu \left( \frac{1}{\sqrt{g}} \partial_\sigma p^\sigma \right) \end{aligned}$$

If there is no source, then we have that  $\frac{1}{\sqrt{g}} \partial_\sigma p^\sigma = \lambda$ , which is constant.

So  $\int \sqrt{g} \nabla_\mu A^\mu \nabla_\nu A^\nu \rightarrow \lambda^2 \int \sqrt{g}$  gives a “dynamical” cosmological term

### 4. Field equation for gravity

- a. The hard part of finding the field equation for gravity is varying  $\sqrt{g}g^{\alpha\beta}R_{\alpha\beta}$ . However, we can use the trick that  $\sqrt{g}g^{\alpha\beta}\delta R_{\alpha\beta}$  is a total derivative.

To prove this relatively painlessly, we can adopt a system of locally geodesic coordinates ( $\Rightarrow \partial_\alpha g_{\beta\gamma} = 0$ ).

Then

$$\begin{aligned} g^{\alpha\beta}\delta R_{\alpha\beta} &= g^{\alpha\beta} \left( \partial_\mu \delta \Gamma_{\alpha\beta}^\mu - \partial_\alpha \delta \Gamma_{\beta\mu}^\mu \right) \\ &= \partial_\mu \left( g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\mu - g^{\mu\beta} \delta \Gamma_{\beta\alpha}^\alpha \right) \\ &\equiv \partial_\mu \omega^\mu \end{aligned}$$

Note that  $\delta\Gamma$  has no inhomogeneous terms in its transformation law - it is a tensor! So  $g^{\alpha\beta}\delta R_{\alpha\beta} = \partial_\mu \omega^\mu \rightarrow \nabla_\mu \omega^\mu$  is now valid in any coordinate system. So  $\sqrt{g}g^{\alpha\beta}\delta R_{\alpha\beta} = \sqrt{g}\nabla_\mu \omega^\mu = \partial_\mu (\sqrt{g}\omega^\mu)$  is a boundary term; it does not contribute to the Euler-Lagrange equations.

Thus with  $S = \kappa \int \sqrt{g}R$  we get

$$\begin{aligned} \delta S &= \kappa \int \left( \delta \sqrt{g} \overbrace{g^{\alpha\beta} R_{\alpha\beta}}^R + \sqrt{g} \delta g^{\alpha\beta} R_{\alpha\beta} \right) \\ &= \kappa \int -\frac{1}{2} \sqrt{g} g_{\alpha\beta} \delta g^{\alpha\beta} R + \sqrt{g} R_{\alpha\beta} \delta g^{\alpha\beta} \\ &= \kappa \int \sqrt{g} \left( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \delta g^{\alpha\beta} \end{aligned}$$

- b. We have for the cosmological term

$$\begin{aligned} \delta S &= -\Lambda \int \delta \sqrt{g} \\ &= \Lambda \int \sqrt{g} \left( \frac{1}{2} g_{\alpha\beta} \right) \delta g^{\alpha\beta} \end{aligned}$$

- c. Matter

$$\begin{aligned} S &= \int \sqrt{g} \Lambda \\ \delta S &= \int \left( \frac{\delta \sqrt{g} \Lambda}{\delta g^{\alpha\beta}} - \partial_\mu \frac{\delta \sqrt{g} \Lambda}{\delta \partial_\mu g^{\alpha\beta}} \right) \delta g^{\alpha\beta} \end{aligned}$$

We define the energy-momentum tensor by

$$\frac{\delta \sqrt{g} \Lambda}{\delta g^{\alpha\beta}} - \partial_\mu \frac{\delta \sqrt{g} \Lambda}{\delta \partial_\mu g^{\alpha\beta}} = \frac{\sqrt{g}}{2} T_{\alpha\beta}$$

We will now see that this makes sense with both examples and conservation laws.

- i. Examples:



- Scalar field:

$$\begin{aligned}\Lambda &= \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2 \\ \frac{\delta\sqrt{g}\Lambda}{\delta g^{\alpha\beta}} &= -\frac{1}{2}\sqrt{g}g_{\alpha\beta}\Lambda \\ \frac{\sqrt{g}}{2}T_{\alpha\beta} &= -\frac{1}{2}\left(\sqrt{g}g_{\alpha\beta}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right)\right) + \frac{1}{2}\sqrt{g}\partial_\alpha\phi\partial_\beta\phi \\ T_{\alpha\beta} &= \partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}g_{\alpha\beta}m^2\phi^2 - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\end{aligned}$$

In flat space we have:

$$\begin{aligned}T_{00} &= \partial_0\phi\partial_0\phi + \frac{1}{2}m^2\phi^2 - \frac{1}{2}\left(\partial_0\phi\partial_0\phi - (\nabla\phi)^2\right) \\ &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \stackrel{\checkmark}{=} WSB\end{aligned}$$

- Maxwell field: (now we use  $\frac{\delta}{\delta g^{\mu\nu}}$ )

$$\begin{aligned}\delta\left(\sqrt{g}g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}\right) &= -\frac{1}{2}g_{\mu\nu}\sqrt{g}g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta} + 2\sqrt{g}g^{\beta\delta}F_{\mu\beta}F_{\mu\delta} \\ T_{\mu\nu} &= \frac{2}{\sqrt{g}}\left(-\frac{1}{4}\right)\delta\left(\sqrt{g}g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}\right) \\ &= -F_{\mu\beta}F_{\nu\delta}g^{\beta\delta} + \frac{1}{4}g_{\mu\nu}g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta} \Rightarrow \\ g^{\mu\nu}T_{\mu\nu} &= 0 \quad (\checkmark)\end{aligned}$$

In flat space we have:

$$\begin{aligned}T_{00} &= E^2 + \frac{1}{4} \cdot 2(B^2 - E^2) \\ &\stackrel{\checkmark}{=} \frac{1}{2}(E^2 + B^2)\end{aligned}$$

As an exercise, check the Poynting vector and stresses for the Maxwell field.

## ii. Conservation laws

Preliminary: We expect the conservation of energy-momentum to be tied up with invariance under translations. So: let us translate!

$$x'^\mu - x^\mu = \delta x^\mu = \epsilon^\mu, \quad \text{small, vanishing at } \infty$$

$$\begin{aligned}g'^{\mu\nu}(x') &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta}(x) \\ &\approx g^{\mu\nu}(x) + \partial_\alpha \epsilon^\mu g^{\alpha\nu} + \partial_\beta \epsilon^\nu g^{\mu\beta} \\ g'^{\mu\nu}(x') &\approx g^{\mu\nu} + \partial_\alpha g^{\mu\nu} \epsilon^\alpha\end{aligned}$$

$$\text{Thus } \delta g^{\mu\nu} = \underset{\text{not } x'}{\underbrace{g'^{\mu\nu}(x)}} = \partial_\alpha \epsilon^\mu g^{\alpha\nu} + \partial_\alpha \epsilon^\nu g^{\mu\alpha} - \partial_\alpha g^{\mu\nu} \epsilon^\alpha$$

Now notice that

$$\begin{aligned}
\delta g^{\mu\nu} &= g^{\alpha\mu} \nabla_\alpha \epsilon^\nu + g^{\alpha\nu} \nabla_\alpha \epsilon^\mu \quad (\text{Killing equations}) \\
&= g^{\alpha\mu} \partial_\alpha \epsilon^\nu + g^{\alpha\mu} \cdot \overbrace{\frac{1}{2} g^{\nu\beta} (\partial_\alpha g_{\beta\rho} + \partial_\rho g_{\alpha\beta} - \partial_\beta g_{\alpha\rho})}^{\Gamma_{\alpha\rho}^\nu} \epsilon^\rho \\
&\quad + g^{\alpha\nu} \partial_\alpha \epsilon^\mu + g^{\alpha\nu} \cdot \frac{1}{2} g^{\mu\beta} (\partial_\alpha g_{\beta\rho} + \partial_\rho g_{\alpha\beta} - \partial_\beta g_{\alpha\rho}) \epsilon^\rho \\
&= g^{\alpha\mu} \partial_\alpha \epsilon^\nu + g^{\alpha\nu} \partial_\alpha \epsilon^\mu + g^{\alpha\mu} g^{\rho\nu} \partial_\rho g_{\alpha\beta} \epsilon^\rho \\
&= g^{\alpha\mu} \partial_\alpha \epsilon^\nu + g^{\alpha\nu} \partial_\alpha \epsilon^\mu - \partial_\rho g^{\alpha\beta} \epsilon^\rho
\end{aligned}$$

where the last step follows from differentiating:  $g_{\alpha\mu} g^{\mu\beta} = \delta_\alpha^\beta$ ; so  $\partial_\lambda g_{\alpha\mu} g^{\mu\beta} + g_{\alpha\mu} \partial_\lambda g^{\mu\beta} = 0$ . Multiplying by  $g^{\rho\alpha}$ , we get  $\partial_\lambda g_{\alpha\mu} g^{\mu\beta} g^{\rho\alpha} = -\partial_\lambda g^{\rho\beta}$ . We can also write this as  $\delta g^{\mu\nu} = \epsilon^{\nu;\mu} + \epsilon^{\mu;\nu}$ .

Now we have, from the invariance of the action and symmetry of  $T_{\mu\nu}$  in its definition

$$0 = \int \sqrt{g} T_{\mu\nu} \epsilon^{\mu;\nu}$$

But also

$$\begin{aligned}
0 &= \int \partial^\nu (\sqrt{g} T_{\mu\nu} \epsilon^\mu) \\
&= \int \sqrt{g} \nabla^\mu (T_{\mu\nu} \epsilon^\mu) \\
&= \int \sqrt{g} \nabla^\nu T_{\mu\nu} + \underbrace{\int \sqrt{g} T_{\mu\nu} \epsilon^{\mu;\nu}}_0
\end{aligned}$$

Since this holds for an arbitrary  $\epsilon^\mu$ , we conclude that  $\nabla^\nu T_{\mu\nu} = 0$ .

## 1.8 Newtonian Limit

We should be able to identify Newtonian gravity (and fix the coupling constant) by looking at the situation with nearly flat space and only  $T_{00} = \rho$  significant. (For now, of course, we ignore the cosmological term.) The stationary action condition gives

$$\begin{aligned}
\kappa \left( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) &= \frac{1}{2} T_{\alpha\beta} \quad \text{or} \\
2\kappa R_{\alpha\beta} &= T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \quad (g_{00} \approx c^2 \gg g_{ij})
\end{aligned}$$

Focus on  $R_{00}$ :

$$2\kappa R_{00} = \frac{\rho}{2}$$

in  $R_{\dots}$  terms with  $\Gamma\Gamma$  pieces higher order. Also terms with  $\frac{\partial}{\partial x^0} \sim \frac{1}{c} \frac{\partial}{\partial x^i}$  are small. Thus in

$$\begin{aligned}
R_{00} &\approx \partial_\gamma \Gamma_{00}^\gamma - \partial_0 \Gamma_{0\gamma}^\gamma \\
&\approx \frac{1}{2} \nabla^2 g_{00}
\end{aligned}$$

and finally

$$2\kappa\nabla^2 g_{00} = \rho$$

To interpret this, write  $g_{00} = 1 + \epsilon$  ( $g^{00} = 1 - \epsilon$ ). The action density of matter is perturbed by

$$\frac{\delta(\sqrt{g}\Lambda)}{\delta g^{00}} \approx -\frac{\rho}{2}\epsilon$$

This looks like the Newtonian coupling if  $\epsilon = 2\phi$ . The equation

$$2\kappa\nabla^2 g_{00} = \rho \Rightarrow 4\kappa\nabla^2 \phi = \rho$$

This fixes  $\kappa = \frac{1}{16\pi G_N}$ .

$$\left( \phi = -\frac{G_N M}{r}; \nabla\phi = \frac{G_N M}{r^2} \hat{\mathbf{r}} \xrightarrow{\text{Gauss}} \int dV \underbrace{\nabla^2 \phi}_{=\frac{M}{4\kappa}} = 4\pi G_N M \right)$$

Note on the appendices and scholia:

These are not necessarily self-contained, specifically, they refer to facts about quantum field theory and the standard model that are not assumed elsewhere in the course. Don't worry if not everything is clear (or even meaningful) to you at this stage. Ask me if you're curious!

Central material

1. The notion of local Lorentz invariance, vierbeins, and  $\mathcal{R}$  recipe (Appendices 1-2).
2. The idea that we are building a model of the world and are free to try anything (Scholium 4).

## Appendix 1: Spinors and Local Lorentz Invariance

Spinor fields play a very important role in fundamental physics, so we must learn how to treat them in general relativity. The essential thing is to define  $\gamma^a$  matrices. They transform under Lorentz transformations (that are in  $SO(3,1)$  not  $GL(4)$ ). Indeed the defining relation

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}$$

refers to the flat-space Minkowski metric and the transformation law

$$\gamma'^a = S^{-1}(\Lambda)\gamma^a S(\Lambda) = \Lambda^a_b \gamma^b$$

requires an  $S$  that does not exist in general. ( $\Lambda \in GL(4)$ .)

So we postulate local Lorentz invariance under transformations of this form with  $\Lambda(x)$  a function of  $x$ ,  $\Lambda(x) \in SO(3,1)$ . To connect this structure to the metric we introduce a vierbein  $e^a_\mu(x)$  such that

$$\begin{aligned} \eta_{ab} e^a_\mu(x) e^b_\nu(x) &= g_{\mu\nu}(x) && \text{("square root" of the metric)} \\ g^{\mu\nu} e^a_\mu(x) e^b_\nu(x) &= \eta^{ab} && \text{("moving frame")} \end{aligned}$$

Now, for example, we can form the Dirac equation

$$(\gamma^a e^a_\mu D^\mu + m) \psi = 0$$

But  $D_\mu$  needs discussion. We want invariance under local Lorentz transformations. This requires (exercise!)

$$D_\mu S(\Lambda(x)) = S(\Lambda(x))D_\mu$$

a typical gauge invariance. We solve this problem “as usual” by introducing a gauge potential

$$\omega_\mu^{ab}(x) \in \mathfrak{so}(3,1)$$

where  $\mathfrak{so}(3,1)$  is the Lie algebra corresponding to  $SO(3,1)$ , and thus  $\omega_\mu^{ab}(x) = -\omega_\mu^{ba}(x)$ , and (in matrix notation)

$$\omega'_\mu(x) = \Lambda^{-1}\omega_\mu(x)\Lambda - \Lambda^{-1}\partial_\mu\Lambda$$

and writing

$$D_\mu = \nabla_\mu + \omega_\mu \cdot \tau$$

where the  $\tau$  matrices provide the appropriate representation of the symmetry, *e.g.*  $\sigma^{ab} = \frac{i}{4} [\gamma^a, \gamma^b]$  for spin  $\frac{1}{2}$  or the identity for spin 1.

To avoid introducing additional structure (*c.f.* Scholium 4) we demand

$$D_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\alpha e_\alpha^a + \omega_\mu^a{}_b e_\nu^b = 0 \quad (1)$$

leading to a unique determination

$$\omega_\mu^a{}_c = -e_c^\nu \partial_\mu e_\nu^a + e_c^\nu e_\alpha^a \Gamma_{\mu\nu}^\alpha$$

## Appendix 2: Moving Frames Method and Recipe

The discussion of Appendix 1 is not as profound as it should be:  $\omega_\mu^{ab}$  should be “more primitive” than  $\Gamma$ . This is worth pursuing, since it leads to a beautiful analogy and useful formulae.

We can eliminate  $\Gamma$  from the defining relation (1) for  $\omega$  by antisymmetrizing in  $\mu \leftrightarrow \nu$ . Thus

$$\partial_\mu e_\nu^a - \partial_\nu e_\mu^a = \omega_\mu^{ac} e_{c\nu} - \omega_\nu^{ac} e_{c\mu}$$

To solve for  $\omega$  we go through a slight rigamarole, reminiscent of what we did to get  $\Gamma$  from  $\nabla g = 0$

$$\begin{aligned} +e_{a\rho} (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) &= \omega_\mu^{ac} e_{a\rho} e_{c\nu} - \cancel{\omega_\nu^{ac} e_{a\rho} e_{c\mu}} \\ -e_{a\mu} (\partial_\nu e_\rho^a - \partial_\rho e_\nu^a) &= -\cancel{\omega_\nu^{ac} e_{a\mu} e_{c\rho}} + \omega_\rho^{ac} e_{a\mu} e_{c\nu} \\ +e_{a\nu} (\partial_\rho e_\mu^a - \partial_\mu e_\rho^a) &= \cancel{\omega_\rho^{ac} e_{a\nu} e_{c\mu}} - \omega_\mu^{ac} e_{a\nu} e_{c\rho} \end{aligned}$$

$$e_{a\rho} \partial_\mu e_\nu^a - e_{a\rho} \partial_\nu e_\mu^a - e_{a\mu} \partial_\nu e_\rho^a + e_{a\mu} \partial_\rho e_\nu^a + e_{a\nu} \partial_\rho e_\mu^a - e_{a\nu} \partial_\mu e_\rho^a = 2\omega_\mu^{ac} e_{a\rho} e_{c\nu}$$

using  $\omega_\mu^{ac} = -\omega_\mu^{ca}$ . Multiplying both sides by  $e^{\rho e} e^{f\nu}$ , we get

$$2\omega_\mu^{ef} = \overbrace{e^{f\nu} \partial_\mu e_\nu^e}^A - \overbrace{e^{f\nu} \partial_\nu e_\mu^e}^B - \overbrace{e_{a\mu} e^{\rho e} e^{\nu f} \partial_\nu e_\rho^a}^C + \overbrace{e_{a\mu} e^{\rho e} e^{f\nu} \partial_\rho e_\nu^a}^C + \overbrace{e^{\rho e} \partial_\rho e_\mu^f}^B - \overbrace{e^{\rho e} \partial_\mu e_\rho^f}^A$$

or

$$\boxed{\omega_\mu^{ef} = \frac{e^{f\nu}}{2} (\partial_\mu e_\nu^e - \partial_\nu e_\mu^e + e_{a\mu} e^{\rho e} \partial_\rho e_\nu^a - (e \leftrightarrow f))}$$

Now we can construct a curvature by differentiating (say) a space-time scalar, which is a local Lorentz vector field

$$(D_\mu D_\nu - D_\nu D_\mu)\phi^a = \mathcal{R}_{\mu\nu}{}^a{}_b \phi^b$$

This leads to

$$\mathcal{R}_{\mu\nu}{}^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b$$

Now you will be delighted (but not too surprised) to learn that this “gauge” curvature is intimately related to the Riemann curvature we had before; indeed

$$R_{\mu\nu\alpha\beta} = \mathcal{R}_{\mu\nu}{}^a{}_b e_{a\alpha} e^b{}_\beta$$

This actually gives the most powerful recipe for computing  $R$ . The technique of introducing frames  $e_\mu^a(x)$  to make the geometry “locally flat” was developed by E. Cartan and is called the moving frame method. It is usually presented in very obscure ways.

### Scholium 1: Structure and Redundancy

Allowing a very general framework and demanding symmetry is an alternative to finding a canonical form that “solves” the symmetry. Thus we consider general coordinate transformations, but postulate a metric to avoid “solving” for local Lorentz frames and then pasting them together.

Vierbeins or moving frames make this much explicit.

Fixing down to a specific frame is gauge fixing in the usual sense.

### Scholium 2: Why General Relativity?

Ordinary spin-1 gauge fields are in danger of producing wrong-metric particles or “ghosts”. This is because covariant quantization conditions (commutation relatives (?) ) for the different polarizations:

$$[a_\mu^\dagger, a_\nu] = -g_{\mu\nu}$$

if normal for the space-like pieces are abnormal for the time-like and vice versa. Gauge symmetry allows me to show the wrong-metric excitations don’t couple (*e.g.* one can choose  $A_0 = 0$  gauge).

Similarly, general covariance/local Lorentz symmetry are required for consistent quantum theories of spin 2.

### Scholium 3: General Relativity vs. Standard Model

In the Standard Model symmetry and weak cutoff dependence (renormalizability) greatly restrict the possible couplings. One requires a linear manifold of fields (*i.e.*  $\phi_1(x) + \phi_2(x)$  is allowed if  $\phi_1(x), \phi_2(x)$  are).

In General Relativity there is still symmetry, and minimal coupling leads to weak cutoff dependence below the Planck scale. However, the fields manifold is not linear ( $g_{\mu\nu}^1 + g_{\mu\nu}^2$  may not be invertible, hence not allowed) and there is a dimensional fundamental coupling. It looks like an effective theory thus, with spontaneous symmetry breaking, *i.e.*  $e_\mu^a = \langle ? \rangle$ , like the  $\sigma$ -model.

#### Scholium 4: Gauge and Dilaton Extensions

The standard assumption of Riemannian geometry is  $\nabla_\alpha g^{\mu\nu} = 0$ . However, we might want to relax this to incorporation additional symmetry. Some important physical ideas have arisen (or have natural interpretations) along this line.

Weyl wanted to unify electromagnetism with gravity. He postulated

$$\nabla_\alpha g_{\mu\nu} = s_\alpha g_{\mu\nu}$$

and the symmetry

$$\begin{aligned} g'_{\mu\nu}(x) &= \lambda(x)g_{\mu\nu}(x) \\ s'_\alpha(x) &= s_\alpha + \partial_\alpha \lambda \end{aligned}$$

This was the historical origin of gauge invariance!

He wanted to identify  $s_\alpha$  with the electromagnetic potential  $A_\alpha$ . That has problems, but the ideas are profound.

Symmetry of the type  $g'_{\mu\nu}(x) = \lambda(x)g_{\mu\nu}(x)$  arises in modern conformal field theory and string theory. It is called “Weyl symmetry”. Given the importance of massless particles, and the idea that massive fundamental particles can acquire mass by spontaneous symmetry breaking, this idea of a scale symmetry retains considerable appeal.

A closely related variant is

$$\begin{aligned} \nabla_\alpha g_{\mu\nu} &= \partial_\alpha \phi g_{\mu\nu} \\ g'_{\mu\nu} &= \lambda g_{\mu\nu} \\ \phi' &= \phi + \lambda \end{aligned}$$

$\phi$ -fields of this sort are called “dilaton”.