Massachusetts Institute of Technology
Department of Physics
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Problem Set 2

1. Show that the number density of dust measured by an observer whose 4-velocity is $\vec{U}$ is given by $n = -\vec{N} \cdot \vec{U}$, where $\vec{N}$ is the matter current 4-vector.

2. Take the limit of the continuity equation for $|\mathbf{v}| \ll 1$ to get $\partial n/\partial t + \partial(nv^i)/\partial x^i = 0$.

3. In an inertial frame $\mathcal{O}$, calculate the components of the stress-energy tensors of the following systems:

   (a) A group of particles all moving with the same 3-velocity $\mathbf{v} = \beta \vec{e}_x$ as seen in $\mathcal{O}$. Let the rest-mass density of these particles be ρ_0, as measured in their own rest frame. Assume a sufficiently high density of particles to enable treating them as a continuum.

   (b) A ring of N similar particles of rest mass m rotating counter-clockwise in the $x - y$ plane about the origin of $\mathcal{O}$, at a radius a from this point, with an angular velocity ω. The ring is a torus of circular cross-section $\delta a \ll a$, within which the particles are uniformly distributed with a high enough density for the continuum approximation to apply. Do not include the stress-energy of whatever forces keep them in orbit. Part of this calculation will relate ρ_0 of part (a) to N, a, δa, and ω.

   (c) Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius a. The particles do not collide or otherwise interact in any way.

4. Use the identity $\partial_\nu T^{\mu\nu} = 0$ to prove the following results for a bounded system (i.e., a system for which $T^{\mu\nu} = 0$ beyond some bounded region of space):

   (a) $\partial_t \int T^{0\alpha} d^3x = 0$. This expresses conservation of energy and momentum.

   (b) $\partial_t^2 \int T^{00} x^i x^j d^3x = 2 \int T^{ij} d^3x$. This result is a version of the virial theorem; it will come in quite handy when we derive the quadrupole formula for gravitational radiation.

   (c) $\partial_t^2 \int T^{00} (x^i x_i)^2 d^3x = 4 \int T^i{}_i x^j x_j d^3x + 8 \int T^{ij} x_i x_j d^3x$. No pithy wisdom for this one.

5. The vector potential $\vec{A} \doteq (A^0, \mathbf{A})$ generates the electromagnetic field tensor via

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

(a) Show that the electric and magnetic fields in a specific Lorentz frame are given by

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla A^0 \ . \end{aligned}$$

Here, ∇ is taken to be the normal gradient operator in Euclidean space.

(b) Show that Maxwell's equations hold if and only if

$$\partial_\mu \partial^\mu A^\alpha - \partial^\alpha \partial_\mu A^\mu = -4\pi J^\alpha \ .$$

(c) Show that a gauge transformation of the form

$$A_\mu^{\text{new}} = A_\mu^{\text{old}} + \partial_\mu \phi$$

leaves the field tensor unchanged.

(d) Show that one can adjust the gauge so that

$$\partial_\mu A^\mu \ = \ 0 \ .$$

Show that Maxwell's equations take on a particularly simple form with this gauge choice. Use the operator $\Box \equiv \partial_\mu \partial^\mu$ to simplify your result.

6. An astronaut has acceleration g in the x direction (in other words, the magnitude of his 4-acceleration, $\sqrt{\vec{a} \cdot \vec{a}}$, is g). This astronaut assigns coordinates $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to spacetime as follows:

   First, the astronaut defines spatial coordinates to be $(\bar{x}, \bar{y}, \bar{z})$, and sets the time coordinate $\bar{t}$ to be his own proper time.

   Second, at $\bar{t} = 0$, the astronaut assigns $(\bar{x}, \bar{y}, \bar{z})$ to coincide with the Euclidean coordinates (x, y, z) of the inertial reference frame that momentarily coincides with his motion. (In other words, though the astronaut is not inertial — he is accelerating — there is an inertial frame that, at $\bar{t} = 0$, is momentarily at rest with respect to him. This is the frame used to assign $(\bar{x}, \bar{y}, \bar{z})$ at $\bar{t} = 0$.) Observers who remain at fixed values of the spatial coordinates $(\bar{x}, \bar{y}, \bar{z})$ are called coordinate-stationary observers (CSOs). Note that proper time for these observers is not necessarily $\bar{t}$! — we cannot assume that the CSOs' clocks remain synchronized with the clocks of the astronaut. Assume that some function A converts between coordinate time $\bar{t}$ and proper time at the location of a CSO:

   $$A = \frac{d\bar{t}}{d\tau}$$

   The function A is evaluated at a CSO's location and thus can in principle depend on all four coordinates $\bar{t}$, $\bar{x}$, $\bar{y}$, $\bar{z}$.

   Finally, the astronaut requires that the worldlines of CSOs must be orthogonal to the hypersurfaces $\bar{t}$ = constant, and that for each $\bar{t}$ there exists an inertial frame, momentarily at rest with respect to the astronaut, in which all events with $\bar{t}$ = constant are simultaneous.

   It is easy to see that $\bar{y} = y$ and $\bar{z} = z$; henceforth we drop this coordinates from the problem.

   (a) What is the 4-velocity of the astronaut, as a function of $\bar{t}$, in the initial inertial frame [the frame that uses coordinates (t, x, y, z)]? (Hint: by considering the conditions on $\vec{u} \cdot \vec{u}$, $\vec{u} \cdot \vec{a}$, and $\vec{a} \cdot \vec{a}$, you should be able to find simple forms for u^t and u^x.)

   (b) Imagine that each coordinate-stationary observer carries a clock. What is the 4-velocity of each clock in the initial inertial frame?

   (c) Explain why $A(\bar{x}, \bar{t})$ cannot depend on time. In other words, why can we put $A(\bar{x}, \bar{t}) = A(\bar{x})$? (Hint: consider the coordinate system that a different CSO may set up.)

   (d) Find an explicit solution for the coordinate transformation $x(\bar{t}, \bar{x})$ and $t(\bar{t}, \bar{x})$.

   (e) Show that the line element $ds^2 = d\vec{x} \cdot d\vec{x}$ in the new coordinates takes the form

   $$ds^2 = -dt^2 + dx^2 = -(1 + g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2.$$