

[SQUEAKING]

[RUSTLING]

[CLICKING]

SCOTT

HUGHES:

The key textbook for this class is Sean Carroll's textbook on general relativity. It's now almost 20 years old. I would say-- I think it's listed on the website as required. I would actually call it sort of semi-required. It is where I will tend to post most of the readings to the course.

It's a really good, complete textbook for a one semester course, which is what we have, and I will not be going through the entire thing. You can't in a one semester course. And I will, from time to time, there will be a few topics that I cannot go as in into as deeply as I would like, and if we had two semesters maybe I'd be able to do a little bit more. And so for those who are interested, I will suggest readings in this. My personal favorite supplement to this is a textbook by Bernard Schutz , *A First Course in General Relativity*.

OK, so these are all things where-- so the MIT bookstore, I'm not sure how much they carry. They're all available through Amazon, and you can definitely find these kind of things. If you get Schutz's textbook, definitely get the second edition. The first edition contains errors.

There is actually a very important geometric object that we are going to introduce in a couple of weeks that his textbook has a really clever derivation of. I remember seeing that and thinking, wow, that's really clever. The reason he was able to do it so simply is because it's wrong.

AUDIENCE:

[LAUGH]

SCOTT

HUGHES:

It's corrected in the second edition. Another one is *Gravitation* by Misner, Thorne and Wheeler. This is sort of a bit of personal history for me. I was Thorne's graduate student, and I used this textbook when I learned this subject originally.

I, frankly, do not recommend this textbook to somebody who is learning the subject

for the first time. It's a good place for a reference for certain things. It's available in the reading room. Know where you can get a copy and pick it up.

So first of all, picking it up is kind of a-- it's good exercise. It's actually like a-- you know, it's a huge book. It gravitates. So it's a great reference and has a couple of good sections in it for new students.

But I will indicate from time to time, especially some of the stuff we do in the second half of the class, my lectures are kind of inspired by Misner, Thorne and Wheeler-- known as MTW. And for those of you who have sort of a more mathematically minded approach to things, *General Relativity* by Wald. In the course syllabus I call this the uber book. This really is sort of the self-contained book that really pins down the subject very, very well. It's quite terse and formal, but also very, very clear. If you are a mathematically minded thinker, you will find this to be a really good textbook to refer to.

There is one particular derivation that I'm going to do in about a month and a half that I essentially take from Wald's textbook because it's just beautiful. Other things in it-- to me, it's a little too terse from my own personal taste, but others may find it to be pretty good.

Two other quick things. So there are 11 problem sets and your grade is determined entirely on these problems sets. The syllabus gives the schedule for when they will be posted and when they are to be handed in. 11 does not divide evenly into 100, so what we do is we have 10 that are worth 9% of your grade and the 11th is worth 10% of your grade.

Is that 11th one really 1% longer? I don't know. Again, I'm taking the viewpoint that you are graduate students, or at least you're playing one for the next hour and a half, and if you're going to sweat the difference of one percentage point on that thing, come on. This is about learning the material, don't worry about those little details that much.

That is where all of your assessment is going to come from. The very first time I lectured this course I had a final exam, and it just turned to be a complete waste of my time and the students' time. You really cannot write-- you either write problems that are so easy that you can do them in your sleep, or they're so difficult you can't

do them in the time period of an exam. So we're just going to stick with problem sets and that's fine.

Let's move into, then, the way the course is going to be structured. So my presentation of this material-- the first half of the course basically up to spring break is essentially the mathematical foundations of general relativity. There are several choices that need to be made when you're doing this.

This often, in some universities, if they have multiple semester sequences, what I'm going to cover in this first half of our semester in some places goes for a full semester. And what this means is there's a couple of things that I just cannot cover in quite as much depth. Those will be things where, for those of you who are interested in it, have that kind of a mathematical thinking of things, happy to push you to additional readings. We can dive in and look at it a bit more depth. My goal is to give you just enough stuff that we can do the most important applications of this subject. And I'm an astrophysicist, and to be blunt, most of my really interesting applications tend to be things that have to do with things like cosmology, black holes, dense stars, and things like that.

And so I want to get enough formalism together that we can get to that part of things. And so the goal of this is that by the week right before spring break what we will do is "derive"-- I put that in quotes, and you'll see why a little bit later-- the Einstein field equations that govern gravity in general relativity. So there are several things that we could do that are not strictly necessary to get there, and just because of time limitations I'm going to choose to elide a few of these topics.

The second half will then be applications. We will use everything we derived in the first half to see how general relativity gives us a relativistic theory of gravity. We will begin applying it.

We'll see how Newton's law is encoded in these field equations. We'll see how we go beyond Newton's law, get some of the classic tests of general relativity, and then start looking at solving it for more interesting systems. Looking at the evolution of the universe as a whole, looking at the behavior of black holes, looking for gravitational waves, constructing the spacetime of neutron stars, things like that. So it's a fun semester. It sort of works well to fit these two things in like this.

And for those of you who are interested in taking things more deeply, there's a lot of room to grow after this. And it does look like-- so people who particularly would like to go a little bit more detail on some of the math, if you've looked at the syllabus, I have one of my absolute favorite quotes from a course evaluation is put on there. Where a student in 2007 or so wrote, "The course was fine as it was, but Professor Hughes as an astrophysicist tended to focus on really mundane topics like cosmology and black holes." If you think those are mundane topics, what can I say, guilty as charged.

But for those you who do want to take a different approach these sorts of things, we'll probably alternate lecturing this course in the future between someone from the CTP who works more in quantum gravity and things related to that. That'll be Netta Engelhardt in spring of 2021.

We're now ready to start talking about, after doing all this sort of prep, we can actually talk about some of the foundations of the theory. So before I dive in, are there any questions? All right.

What we're going to begin doing for the first couple of weeks-- well, not really first couple weeks. The first couple of lectures, is we're going to begin by discussing special relativity, but we're going to do special relativity using mathematical language that emphasizes the geometric nature of this form of relativity. What this does is it allows us to introduce basically the formalism, the notation, all the different tools that are important for when things get more complicated.

When we apply a lot of these tools to special relativity, like we will be doing in the first three or so lectures of this course, it's kind of like swatting a mosquito with a sledgehammer. You really don't need that much mathematical structure to discuss special relativity. But you're going to be grateful for that sledgehammer when we start talking about strong field orbits of rotating black holes, right. And so the whole idea of this is to develop the framework in terms of a physical system where it's simple to understand what is going on. These are sort of a way to introduce the mathematical tools in a place where the physics is straightforward and then kind of carry forward from there.

I will caution that as a consequence of this many students find these first three

lectures to be a little on the dull side. So, sorry. It's just some stuff that we kind of have to get out there, and then as we generalize to more interesting mathematical objects, more interesting physical settings, it gets more interesting.

All right, so let's dive in. So the setting for everything that we will be doing is a geometric concept known as spacetime. So we give you a precise mathematical definition of spacetime. A spacetime is a manifold of events that is endowed with a metric.

So, a wonderful mathematical definition, and I've written it in a way that requires me to carefully define three additional terms. The concepts that I've underlined here, I've not defined them yet precisely exactly what I mean by them. So let's go over to the sideboard and talk about what exactly these are.

So a manifold-- if you are a mathematician, you might twitch a little bit about the way that I am going to define this, and I will point you to a reading that does it a little bit more precisely. But for the purposes of our class, a manifold is essentially just a set or a collection of points with well understood connectedness properties.

What I mean by that, is I'm going to talk about manifolds of space and time. I haven't defined an event yet, but I'm about to, but I'm going to say that there's a bunch of events that happen at this place and at this time and a bunch of events that happen at this place and at this time. And the manifold, the spacetime, gives me some notion of how I connect the events over here to the events over here. A manifold is a tautological concept. It's all about how one connects one region to another one.

So if you're working on a manifold that lives on the surface of a donut, you have a particular topology associated with that. If it lives on the surface of a sphere you have a different topology associated with it. If you would like to see more careful and more rigorous discussion of this, this is one of the places where Carroll is very good. So go into Carroll, at least in the edition that I have it's on pages 54 to 62. He introduces a bit of additional mathematical machinery, discusses things with a little more rigor than I'm doing here-- this isn't rigorous at all, so significantly more rigor than I'm doing here. Those of you are into that, that should be something that you enjoy.

So an event. This is when and where something happens. Could be anything. From our point of view, the event essentially is going to be the fundamental notion of a coordinate in space time.

We will actually, in many cases-- actually, that's bad word choice, I should say. Coordinates are actually sort of labels that we attach to events. We are going to be free to adjust those labels, but the underlying geometric idea that the event is here, that's independent of the coordinates that we choose. So we will label these things with coordinates. But the event itself exists independent of these labels.

Just to give an example, there might be one event which is I punch myself in the head. And so I'm very egotistic, so I will say this event happened at time 0-- x , y , and z equals 0. Because I define this corner of my skull as the origin the coordinate system, and I always think whatever's happening right now is the origin of time.

Those of you out in the room are also egocentric and you would say whatever, I'm going to call that-- let's say you are at y of 3 meters and I'm going to put the floor as the origin of my z -axis, so z of 1.7 meters or whatever. You will come with your own independent labeling of these things. So you're all familiar with the idea that we can just change coordinate systems. I'm going to harp on this a bit, though, because there's going to be a really important distinction we make between geometrical objects that live in this manifold of spacetime and how we represent them using labels that might be attached to coordinate systems. And I'm going to come back to this when we start talking about some additional geometric objects in just a couple of minutes.

So the last object that I have introduced into here is one that we will begin talking about in a lot more detail in the next lecture, but let me put it into here right away. And so that is the metric. Metric comes from a root meaning to measure, and what this is is it's something it gives me a notion of distance between events in the manifold.

For physics to work, this has to exist, right? But it's worth knowing that the idea of a manifold is in some way more primitive than this. You can have a manifold without any notion of a metric attached to it. And if that's the case-- people like to joke that if you don't know what the difference is between a metric with a manifold and

without a man-- excuse me, a manifold with a metric and without a metric, feel free to drink coffee out of a donut. Because topologically, those are the exact same thing, but their geometry, which is encoded in the metric, which tells me how the different points on that manifold are arranged, are rather different.

What this basically does is it's going to give me a mathematical object that enforces or really conveys the idea that different events in this manifold have a particular distance between them. So without this, a manifold has no notion of distance encoded in it. So the two things together really make this concept come to life. You can see a lot more, like I said. You can get more information about many of these concepts from the readings in Carroll, Wald's textbook also goes into quite a lot of detail about this stuff.

So this is the venue. This is the setting in which we are going to talk about things. And just cutting forward roughly 2 and 1/2 months' worth of lectures, what we are going to find is that part of Einstein's genius is that it turns out that this notion of the metric ends up encoding gravity. So that's kind of where we're going to end up going with things.

The idea that the mathematical structure that tells me how far apart two events are is intimately connected to the properties of gravity. It's pretty cool, and it is something that-- physics is an experimental science. All of our measurements are consistent with it. So that's cool.

So everything I have said so far, nothing but math. Nothing but definitions. So let's start working with a particular form of physics. So we are going to begin, as I said, with special relativity.

This is the simplest theory of spacetime that is compatible with physics as we know it. Not fully compatible, but does a pretty good job. And we'll see that it turns out to correspond to general relativity when there is no gravity.

So to set this up we need to have some kind of a way of labeling our events. And so I'm going to introduce kind of a conceptual-- you almost think of it as scaffolding, which we're going to use to build a lot of our concepts around. And in this one-- I mean that in a kind of an abstract sense. There's going to be sort of the edifice that we use to help us build the building that's going to be the mathematics of general

relativity.

But this one really is kind of like a scaffolding. Because what I want to introduce here is a notion of what is called an inertial reference frame. So I'm going to sketch this quickly, and I'm going to post to the course website a chapter from an early draft textbook by Roger Blandford and Kip Thorne.

So when I talk about the inertial reference frame, I want you to sort of visualize in your head a lattice of clocks-- clocks and measuring rods-- that allows us to label, in other words, to assign coordinates any event that happens in spacetime. So just sort of, in your head, imagine that there's this grid of little clocks and measuring rods, and a mosquito lands on your head. It's right near a particular rod and a particular clock. It bites you, that's an event. You slap it, that's another event.

And the measuring rods and events are what allows you to sort of keep track of the ordering of those events and where they happen in this four dimensional manifold of spacetime. So I'm going to require this lattice to have a certain set of properties. First, I'm going to say that this lattice moves freely through spacetime.

What do I mean by moving freely? I mean no forces act on it, it does not rotate, it is inertial. Every clock and every measuring rod has no inertia, no force is acting on it at all.

You look at that, you might think to yourself, well, why don't you make it at rest? Well, I did. It's at rest in respect to someone, but we might have a different observer who's coming along who has no forces acting on her, and she's moving relative to me at three quarters the speed of light. It's not at rest with respects to that observer. That's actually kind of the key here.

So this inertial reference frame is at rest with respect to someone who feels no forces but not to all observers. I'm going to require that my measuring rods are orthogonal to each other. So they define an orthogonal coordinate system, and I am also going to require that the little markings on them that tell me where things happen are uniformly ticked. In other words, I'm going to just make sure that the spacing between tick marks here is exactly the same as the spacing between tick marks here.

You may sort of think well, that's a result or an idea worthy of the journal *Duh*, but it's important to specify this. You want to make sure that the standard you are using to define length is the same in this region of spacetime as it is over in this region of spacetime. When we start getting into general relativity, we start to see there can be concerns about this coming about, so it's worth spelling it out and making it clear at the beginning.

I'm also going to require that my clocks tick uniformly. We're going to make this lattice that fills all of spacetime using the best thing that Swiss engineers can make for us, we want to make sure that one second, an interval of one second, is the same here in this classroom as it is somewhere off in the Andromeda Galaxy. We want to make sure that there is no evolution to the time standard when we do this.

Finally, I'm going to synchronize all these clocks with each other in the following way. This is going to use the Einstein synchronization procedure. This is the first place where a little bit of physics is actually beginning to finally enter our discussion. I'll comment that-- I'm going to go through what this procedure is in just a second.

But an Easter egg here. Whenever you see a name that has Einstein in it, your ears should perk up a little. Because it probably means this is something important. This is, after all, a course in relativity, and that tends to be the things that end up mattering. Even when they end up being really easy to-- things we look at now and kind of see as fairly obvious, it's important, OK, and we often attach Einstein's name to this.

So this Einstein synchronization procedure, this takes advantage of the fact-- and by the way, we're not going to teach special relativity in this course, 8.962. I assume you have already studied special relativity and you're all experts in this, and so I can freely borrow from its important results. This procedure takes advantage of the fact that the speed of light is the same to all observers, no matter what inertial reference frame they might be in.

So the speed of light is a key invariant. It connects-- because it's a speed, it connects space and time, and because it is the same to all observers, it defines a particular standard for relating space and time that is going to have important invariant meaning associated with it. Just to remind you what this means-- let's see,

do I have a laser pointer with me? I might, but don't worry about it. I have a pretend laser pointer. My chalk's a laser pointer.

I point my laser pointer at the wall and you all see it dashing across the room at 300,000 kilometers per second. I then start jogging at half the speed of light, as is my wont, and continuing to point that, you guys measure the light going across the room. You still measure 300,000 kilometers per second.

I, on the other hand, measure the light coming out of my laser pointer and I get 300,000 kilometers per second. So just because my laser pointer is moving at half the speed of light according to you doesn't mean the light that's coming out of it is boosted to a higher speed. If you studied special relativity you'll know that its energy is boosted to a higher energy level, but the speed of light is always just c . So we're going to take advantage of that to come up with a way of synchronizing our clocks.

The way it works is this. So let's look at a two dimensional slice of my lattice here. Time, and let's make this be the x-axis. And so I will have, let's say, this is where clock one exists and this is where clock two exists.

Let's go into the reference frame that is at rest with respect to this lattice. We want to synchronize clock one with clock two. So as time marches on, these guys stand still. So here's the path in spacetime-- the world line, traced out by clock one. Here's the clock path in spacetime, the world line, traced out by clock two.

So let's say-- let's call this event-- let's say that this event happens at a time t_{1e} . t_{1e} is when clock one emits a pulse of light. So this light will just follow a little trajectory a bit through spacetime. This goes out and strikes clock two at which point it is bounced back to clock one.

Let's ignore this point for just a second. So let's just say for the moment that this is then reflected back and then it is received back at clock one at a time t_{1r} . Let's make this a little bit neater. t_{1e} , t_{1r} .

Clock one receives the reflected pulse. So the moment at which it bounces, we'll call that t_{2b} , that is the moment at which the light bounces off of clock two. And the way we synchronize our clocks is just by requiring that this be equal to the average

of the emission and the reception time.

Totally trivial idea, right? All I'm saying is I'm going to just require that in order for clock one and clock two to be synchronized to one another, let's make sure that when I bounce light between any two pairs of clocks they are set such that when the light bounces it's the midway point between the total light-- the halfway of the total light travel time that the light moves along. Very, very simple concept.

So there's nothing particularly deep here, but notice Einstein's name is attached to this. And I don't say that to be-- I'm not being sarcastic. It really points to the fact that we are using one of the most fundamental results of special relativity in designing how these clocks work in this inertial reference frame.

Believe it or not, this thing, this really simple concept, it comes back to sort of bite us on the butt a little bit later in this course. Because when we throw gravity into the mix, we're going to learn that gravity impacts the way light travels through spacetime. We're going to get to some objects where gravity is so strong that light cannot escape them. And we're going to find that our perhaps most naive ways of labeling time in the spacetime of such objects kind of goes completely haywire.

Fundamentally, the reason why time is going haywire when you have really strong gravity is because we use light as our tool for synchronizing all of our clocks, and if the way that light moves in your space time is affected, the way you're going to label time is going to be affected. So this is a very simple concept. Again, I sort of emphasize these first couple lectures you're swatting a mosquito with a sledgehammer, but we're setting up this edifice because this will come back and it's important to bear this in mind when things get a little more interesting later in the course. So let's start setting up some geometrical objects here.

Pardon me, let me do one other thing really quickly before I start setting up some geometrical objects. So when I sketched this thing called a spacetime diagram here, I should have talked a little bit about the units that I'm going to use to describe the ticking of my clocks and the spacing of tick marks on my measuring rods. What we will generally do in this course is choose the basic unit of length to be the distance light travels in your basic unit of time.

While you parse that sentence, what that's basically saying is suppose I set my clock

so that they tick every second. Well, if my clocks tick once per second, then my basic unit of length will be the light second. Do you want to put this into more familiar units? That's about 300,000 kilometers.

One of my personal favorites-- if the time unit is 1 nanosecond, the length unit is, of course, one I will call it LNS-- Light Nanosecond. Students who are in 8.033 with me are not allowed to answer this question, does anyone know what one light nanosecond is? Actually, this is a little bit ridiculous. But to within far greater than a percent accuracy it is one foot.

The English unit that comes on these asinine rulers that those of us educated the United States learned in all of our European friends sneer at us about. The speed of light is to incredible precision 1 foot per nanosecond. To be fair, let's make that a wiggly equal.

So what this means is that in the units that I'm going to be working with, if I then want to express the speed of light in these units-- So C is one light time unit per time unit, which we are just going to call 1. So we will generally set the speed of light equal to 1. Just bear in mind what this essentially means is that you can think of, if you want to then convert to your favorite meters per second, furlongs per fortnight, whatever it is that you're most comfortable with, C is effectively a conversion factor then.

And so what this means is that when we do this all velocities that we measure are going to be dimensionless. Really what we're doing is we're measuring them as fractions of the speed of light. Now, with this system of units defined, let's talk about a geometric object.

So let's imagine that O is an observer in the inertial reference frame that I defined a few moments ago. This is a mouthful to say, it's even more of a mouthful to write, so I'm going to typically abbreviate this IRF. So o observes two events, which I will label P and Q . I'll just go to a clean board for this.

So let's say here in spacetime-- so imagine I've got coordinate axes that have been drawn, it's going to be three dimensional, for simplicity I'm not going to actually write them out. Let's say I've got event P here and event Q over here. Now if we were just doing Euclidean geometry in three space, you guys have all known that

once you've got two events written down on a plane or in a three dimensional space, something like that, you can define the displacement vector from one to the other. We're going to do the same thing in spacetime.

So let's call Δx and I'm going to make a comment on notation in just a moment-- this is the displacement in spacetime from P to Q. We're going to define the components of this displacement vector as seen by O. So when I write equals with a dot on it that means the geometric object that I've written on the left hand side is given according to the specified observer by the following set of components, which I'm about to write down. So this looks like so.

So, couple things that I want to emphasize that I'm introducing here, bits of notation that we're going to use over and over again in this term. Notice I am using an over arrow to denote a vector in spacetime. Different texts, different professors use slightly different notations for this.

Those of you who took 8.033 at MIT with Sal Vitale he preferred to write a little under tilde when he wrote that. For us working in four dimensional space time, it is going to be of paramount importance to us, and so we're going to use this over arrow which you probably have all seen for ordinary three dimensional vectors. For us it's going to represent a four dimensional vector.

Now in truth we're not going to use it all that much after the first couple of weeks of the class, our first couple lectures even. Occasionally we'll bust it out, but we will tend to use a more compact notation in which we say Δx , that displacement factor has the components Δx^μ , where μ lies in either t, x, y, and z, or 0, 1, 2, 3. When we set up a problem, we need to make a mapping to what the numerical correspondences between-- I need to tell you that μ equals 0 corresponds to time, and μ equals 1 corresponds to x. We'll switch to other coordinate systems and I'll have to be careful to say, almost always, μ equals 0 will be time.

But what the other three correspond to, that depends on the coordinate system. It might be a radius, it might be a different angle, some things like that. Again, just sort of being a little overly cautious and careful defining these.

I will note, though, that generally Greek indices in most textbooks, they tend to be

used to label spacetime indices. And then there are times when you might want to just sort of imagine you've chosen a particular moment in time, and you want to look at what space looks like at that time. And so you might then go down to Latin indices to pick out spatial components at some moment in time.

We'll see that come up from time to time, just want you to be aware there is this distinction. And as you read other textbooks, there are a few others that are used. Always just check, usually in some of the introductory chapters of the textbook, they will define these things very carefully.

Wald is an example of someone who actually does anything a little bit different. He tends to use lower case Latin letters from the top of the alphabet to denote spacetime indices, and those from i, j, k , he uses them to denote Latin indices. If you're old enough to get this, this is often called by some of us who grew up in the Dark Ages it is sometimes called the Fortran convention. If you've ever programmed in Fortran, you know why that is. If you didn't, please don't bother learning it, it's really not worth the brain cells it would take.

So we've got this geometric object that is viewed by observer o . Let's now think about what this looks like from the viewpoint of a different observer. A different inertial observer.

Let's say somebody comes dashing through the room here, and observer o sees them running across the room at something like 87% of the speed of light. You know, since, as I have assumed, you are all experts in special relativity, that they will measure intervals of time and intervals of space differently than observer o does.

So here's event P . Here's event Q . Here is Δx . This is all as measured by observer o bar.

Something which I really want to strongly emphasize at this point is that this P , this Q , and this Δx , notice I haven't put bars on any of them. I haven't put primes or anything like that. It is the exact same P and Q and Δx as this over here. That is because P , Q , and Δx are geometric objects whose meaning transcends the particular inertial reference frame used to define the coordinates at which P exists, at which Q exists, and that then defined the Δx .

These geometric objects exist independent of the representation. If I can use an intuitive example-- if I take and I hold-- let's be careful the pose I do with this-- let's say I stick my arm out. I say that my arm is pointing to the left, right?

You guys will look at this and say your arm is pointing to the right, because you're using a slightly different system of coordinates to orient yourself in this room. We're both right. We have represented this geometric object, my arm, in different ways. But me calling this pointing to the left and you calling it pointing to the right doesn't change the basic nature of my arm. It doesn't mean that my blood cells changed because of something like this happening. This has an independent existence.

In the same way, this Δx , it is the displacement, these two events. This might be mosquito lands on my head, this might be me smacking it with my hand, or flying near my head and me smocking with my hand. Those are events that exist independent of how we choose to represent them. So the key thing is we preserve that notion of the geometric object's independent existence. What does change is the representation that the two observers use.

So I'm going to jump to this Greek index notation. And so what I'm going to say is that according to observer \bar{o} , they are going to represent this object by a collection of components that are not the same as the components that are used by observer o . And to keep my notation consistent, let's play a little \bar{o} underneath this arrow. So this is just sort of shorthand for Δx is represented according to \bar{o} by those components. Δx is represented according to o by these components.

And again, since I'm assuming you all are experts in special relativity, we already know how to relate the barred components to the unbarred components. They're related by a Lorentz transformation. So what we would say is Δx is zero bar component, or t bar, if you prefer. It's given by γ -- I will define γ in just a moment, you can probably guess. So I'm imagining an observer that is just moving along the x -axis or the coordinate one axis.

So what went on in that transformation I just wrote down is \bar{O} moves with v along spatial axis 1 with speed v as seen by O . And of course γ is 1 divided by square root of 1 minus v squared. Remember, speed of light is 1 .

We don't want to be writing this crap out every time we have to transform different

representations, so we're going to introduce more compact notation for this. So we're going to say Δx^μ -- this is what I get when I sum over index ν from 0 to 3 of $\Lambda^\mu{}_\nu \Delta x^\nu$. So that's defining a matrix multiplication, and you can read out the components of this Λ matrix from what I've got over there.

And even better yet, Δx^μ is $\Lambda^\mu{}_\nu \Delta x^\nu$. So in this last line, if you haven't seen this before, I am using Einstein's summation convention.

If I have an index that appears in one geometric object in the downstairs position and an adjacent geometric object in the upstairs position and it's repeated-- so repeated indices in the upstairs and downstairs position are assumed to be summed over their full range, from 0 to 3. We're going to talk about this a little bit more, what's going on with this, after I've built up a little bit more of the mathematical structure. In particular, what is the distinction between through the upstairs and downstairs positions.

Some of you might be saying, well, isn't one way of writing it what we call a covariant component and wasn't one a contravariant component? If you know those terms, mazel tov. They're not actually really helpful, and so I kind of deliberately like to use this more primitive wording of calling it just upstairs and downstairs.

Because what we're going to find, the goal of physics, is to understand the universe in a way that allows us to connect this understanding to measurements. And measurements don't care about contravariant versus covariant, and all these things are essentially just ways of representing objects with our mathematics that is sort of a go between from some of our physical ideas to what can eventually be measured. So covariant, contravariant, eh, whatever. At the end of the day, we're going to see as we put these sort of things together it's how these terms connect to one another that matters. The name is not that important.

I do want to make one little point about this as I move forward. It is sort of worth noting that if I think of how I relate the displacement components according to my barred observer relative to those as measured by my unbarred observer, I can think of this Lorentz transformation matrix as what I get when I differentiate one

representation's coordinates with respect to the other representation's coordinates. Kind of trivial in this case, and when we are doing special relativity there is a particular form of the Lorentz transformation that we tend to use, but I just want to highlight this because this relationship between two different representations of a reference frame is going to come up over and over and over again.

This is a more general form, this idea that you are essentially taking the derivative. You're looking at how one representation varies according to the other representation. So using that to think about how to move between one inertial reference frame to another is going to be very important to us. This is a more general form.

I want to make one further point, and then I will introduce a more careful definition of what is meant by a vector and I think that will be a good place for us to stop. So when I look at this particular form-- so let's look at the last thing I wrote there, where I use the Einstein summation convention. This is just a chance for me to introduce a little bit more terminology in notation.

So when I wrote down the relationship $\delta x_{\bar{\mu}} \text{ with } \lambda_{\bar{\nu}} \mu_{\bar{\nu}}$, how I labeled the index that I was summing over is kind of irrelevant. This is exactly the same as $\lambda_{\bar{\mu}} \alpha \delta x_{\alpha}$. I can switch to something else if you have enough fonts available. You can use smiley faces as your index, whatever. The key bit is that as long as you are summing over it, it's kind of irrelevant how you label it.

I think that looks silly, so I'm going to go back to alpha.

When I have an index like that that is being summed over, it's going to sort of disappear at the end of my analysis. Its only role is to serve as a place holder. It allows me to keep things lined up properly so that I can do a particular mathematical operation. So in this equation, ν or α is called a dummy index.

Now, when I'm doing it with something like this, where I'm just relating one set of one indexed objects to another set of one indexed objects, it's kind of trivial to move these things around. We're going to make some much more complicated equations later in this class, and we'll find in those cases that sometimes it's actually really useful to have the freedom to relabel our dummy indices. It allows us to sort

of pick out patterns that might exist among things and see how to simplify a relationship in a really useful way.

On the other hand, I do not have the freedom to change that μ bar that appears there, right. I have to have that being the same on both sides of the equation. Because I'm free to mix around things where it's just going to be summed over and not play a role, but in this my μ bar-- pardon me for a second-- μ bar is not a dummy index, and so I do not have that freedom. We sometimes call this the free index.

As I write that down, it seems like a bit of a strange name, actually, and it's really not free. You're actually constrained in what it can be. What can I say, probably there's some history there that I don't know about.

So let's do our last concept for the day. So let's carefully define a spacetime vector. So a spacetime vector is going to be any quartet of numbers, those numbers we will call components, which transforms between inertial reference frames like the displacement vector.

So if I represent some spacetime vector A as some collection of numbers that as observed by o has components A_0, A_1, A_2, A_3 . If a second inertial observer relates their components to these by-- if that describes the components for the observer o bar, then it's a vector. If you have taken any mathematics, they carefully defined vector spaces, this should be familiar to you. It's a very similar operation to what is done in a lot of other kinds of analysis. The key to making this notion of a vector a sensible one is this transformation law.

So if I had a quartet of numbers, which is say, the number of batteries in my pocket, the number of times my dog sneezed this morning, how many toes I have on my left foot, and-- I'm sick of counting so let's just say 0 for the fourth component-- that does not transform between reference frames by a Lorentz transformation. It's just a collection of random numbers. So not any old quartet of numbers will constitute the components of a vector. It has to be things that have a physical meaning that you connect to what you measure by a Lorentz transformation.

For it to be a good vector, it also has to-- that A has to obey the various linearity laws that define a vector space. So if I had two vectors and I add them together,

then their sum is a vector. If I have a vector and I multiply it by some scalar-- by the way, you have to be a little bit careful when I say scalar here, because you might think to yourself something like, you know, the mass of my shoe, that's a scalar. But you be careful when you're talking about things in relativity, whether the scale you're dealing with is actually a quantity that is Lorentz invariant. With quantity like mass, if you're talking about rest mass-- we'll get into the distinction among these things a bit later-- OK, you're good.

You just have to be careful to pick out something that actually is the same according to all observers. So when I say scalar, this means same to all observers. If this is the case, then I can define D to be that scalar times a vector and it is also a vector. Question? Oh, stretching, OK.

I think, actually, I'm going to stop there. There is one topic that I just don't feel like I can get enough of it for it to be useful right now, so I think yes. I'm going to stop there for today.

When we pick it up on Thursday we're going to wrap up this discussion of vectors-- again, I want to kind of emphasize, I can already see in some places you're getting the 1,000 yard stare. There's no question we're being excessively careful with some of these definitions. They are very straightforward, there's nothing challenging here. But there will be a payoff when we do get to times where the analyses, the geometries we're looking at, get pretty messed up. Having this formal foundation very carefully laid will help us significantly.

So all right, I'm going to stop there for today, and we will pick it up on Thursday.