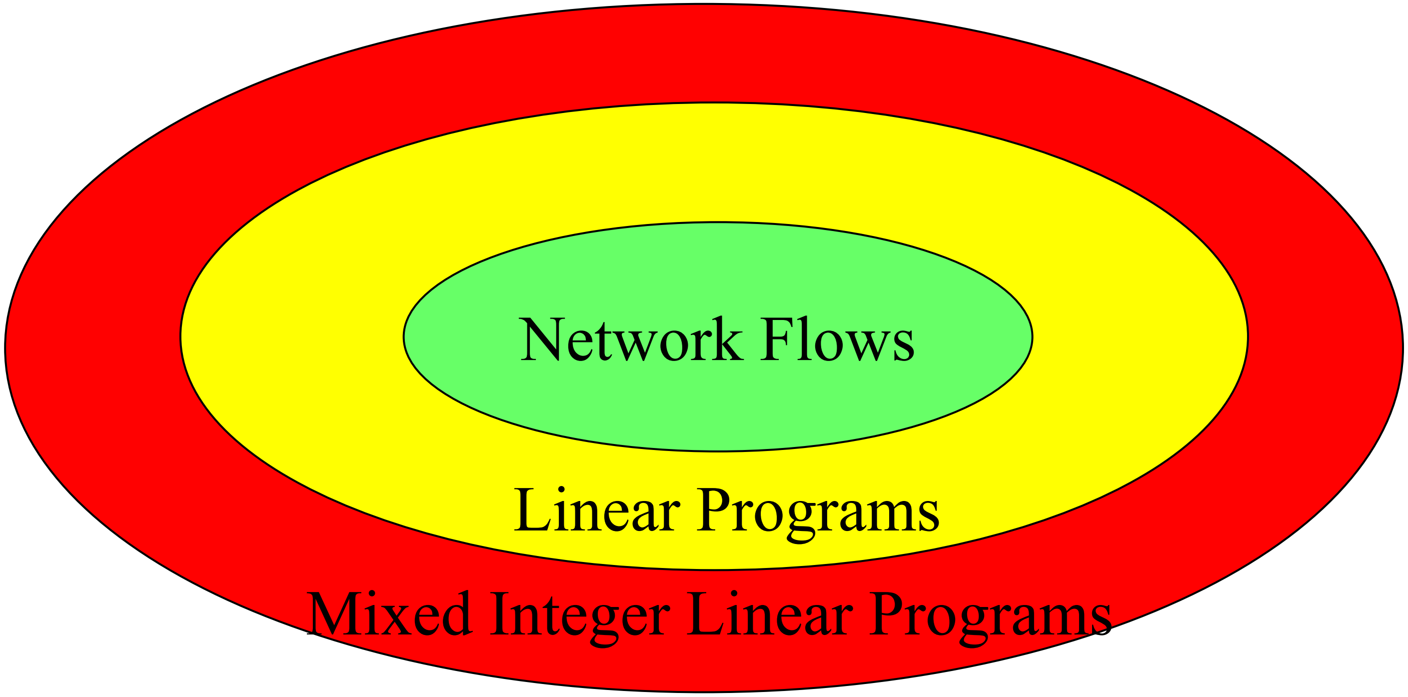
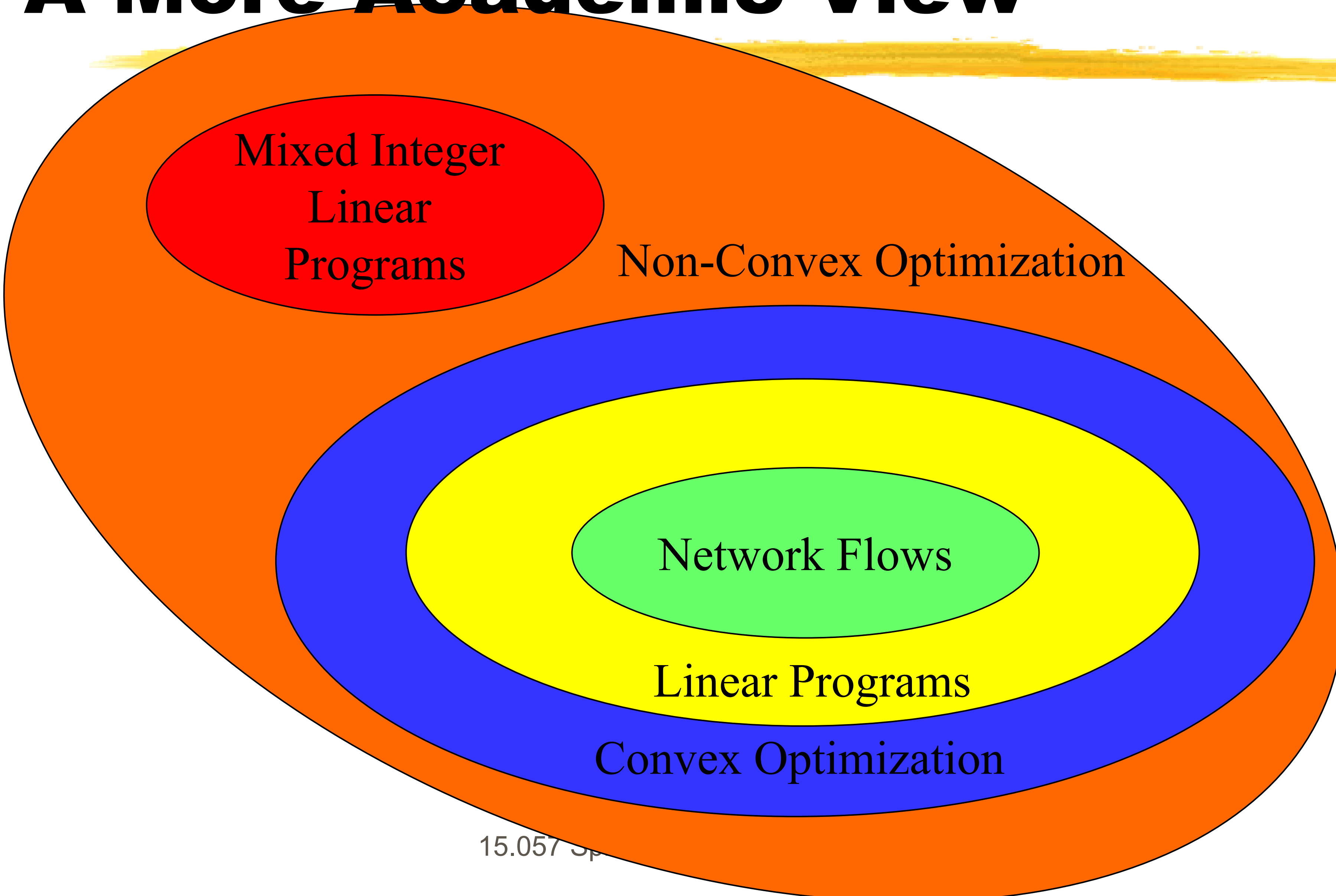
Linear Programming

Hierarchy of Models
Define Linear Models
Modeling Examples
in Excel and AMPL

Hierarchy of Models



A More Academic View



The Differences

	Objective Function	Variables	Constraints
Network Flows	Linear	Continuous	Special Linear Forms
Linear Programs	Linear	Continuous	Linear Forms
Convex Optimization	Convex	Continuous	Convex
Mixed Integer Linear Programs	Linear	Discete or Continuous	Linear Forms
Non-linear Optimization	General	Continuous	General

Our Focus:

- Linear Programs (LP),
- Mixed Integer Linear Programs (MIP)
- Heuristics

Agenda for LP

- First Example
- What is Linear?
- Several Illustrative Examples
  - Excel and AMPL
- Revenue Optimization Application
- Portfolio Optimization

A First Example

Simplified Oak Products Model

Chair Style	Captain	Mate				
Profit/Chair	$ 56	$ 40	Profit			
Poduction Qty.	0	0	0			
	Chair Component		Total Usage		Start Inventory	End Inv.
Long Dowels	8	4	0	≤	1280	1280
Short Dowels	4	12	0	≤	1600	1600
Legs	4	4	0	≤	760	760
Heavy Seats	1	0	0	≤	140	140
Light Seats	0	1	0	≤	120	120
			Chairs		Min. Production	Slack
Chair Production	1	1	0	≥	100	-100

Challenge

- Build a Solver Model

A First Example

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The Model

- Objective: Maximize Profit
  - =SUMPRODUCT(UnitProfit,Production)
  - =$56*Production of Captains + $40*Production of Mates
- Variables: Production
  - $B$4:$C$4
  - Production of Captains
  - Production of Mates

Constraints

- Constraints:
  - TotalUsage <= StartInventory
    - =SUMPRODUCT(LongDowelsPerChair,Production) <= 1280
    - =SUMPRODUCT(ShortDowelsPerChair,Production) <= 1600
    - ...
  - TotalProduction >= MinProduction
    - =SUM(Production) >= 100
- Options
  - Assume Non-negative
  - Assume Linear Model

What's a Linear Model

- What is a linear function?
  - Sum of known constants * variables
  - NOTHING ELSE IS LINEAR
  - Examples:
    - Sum across a row of variables
    - Sum down a column of variables
    - $56*Production of Captains + $40*Production of Mates
  - In Excel
    - SUMPRODUCT(CONSTANTS, VARIABLES)
  - In AMPL
    - sum {index in Index Set}parameter[index]*variable[index]
    - Index Set cannot depend on values of variables

A Test

- Variables: x and y
- Which are linear?
  - $x^2 + y^2$
  - $(1-\text{sqrt}(2))^2 x + y/200$
  - $|x - y|$
  - $x*y$
  - $10/x$
  - $x/10 + y/20$
  - $\text{sqrt}(x^2 + y^2)$

Linear Programs

- Objective:
  - A linear function of the variables
- Variables:
  - May be restricted to lie between a lower bound and an upper bound
  - In AMPL
    - var x >= 1, <= 200;
- Constraints:
  - Linear Function of the variables $\begin{matrix} \leq \\ \geq \\ = \end{matrix}$ Constant

Why These Limitations!

- Can anything real be expressed with such limited tools?
- What do we get for all the effort?

Power of Expression

- The Marketing Hype:
  - LOTS
  - You will be amazed…
  - Call before midnight tonight and get...
- Experience:
  - Most of Almost Everything
  - All of Almost Nothing

My Own Perspective

- Linear Programming
  - Large portions of most real applications
  - Basis for understanding
  - Background for MIP (Mixed Integer Programming)
    - Everything can be modeled with MIP, but...

What do we get for playing?

- Guarantees!
  - Readily available algorithms that
    - Find a provably best solution
    - Quite fast even for large problems
  - Less compelling generally
    - Sensitivity Analysis (not available with MIP)

Review of Sensitivity

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Review of Sensitivity Analysis

Microsoft Excel 8.0a Sensitivity Report
Worksheet: [07OakProductsLP.xls]Sheet1
Report Created: 12/19/01 4:52:52 PM

If the unit profit on the Mate were to drop, how much could it drop before we would quit making it?

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$4	Poduction Qty. Captain	130	0	56	24	16
$C$4	Poduction Qty. Mate	60	0	40	16	12

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$6	Long Dowels Total Usage	1280	4	1280	40	180
$D$7	Short Dowels Total Usage	1240	0	1600	1E+30	360
$D$8	Legs Total Usage	760	6	760	72	40
$D$9	Heavy Seats Total Usage	130	0	140	1E+30	10
$D$10	Light Seats Total Usage	60	0	120	1E+30	60
$D$12	TotalProduction	190	0	100	90	1E+30

More Examples

- Illustrate "tricks"
- Build experience
- AMPL Examples

Blending Example

Eastern Steel Blending Example (Described in Moore et al. Page 105 and following)

		Mines						
		1	2	3	4			
Tons of Ore/Ton of Alloy								
	Cost/Ton	$ 800	$ 400	$ 600	$ 500			
		Lbs of each basic element/Ton of Ore						
	Basic Element					Lbs/Ton of Alloy		Min. Lbs/Ton of Alloy
	A	10	3	8	2	0	≥	5
	B	90	150	75	175	0	≥	100
	C	45	25	20	37	0	≥	30

Challenge

- Build a Solver model

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Basic Element							
A	10	3	8	2	0	≥	5
B	90	150	75	175	0	≥	100
C	45	25	20	37	0	≥	30

AMPL Model

```
set MINES;
set ELEMENTS;

param MinLbs{ELEMENTS};
param CostPerTon{MINES};
param LbsPerTon{MINES, ELEMENTS};

var Tons{MINES} >= 0;

minimize TotalCost:
	sum{mine in MINES} CostPerTon[mine]*Tons[mine];
s.t. CompositionConsts {elem in ELEMENTS}:
	sum{mine in MINES} LbsPerTon[mine, elem]*Tons[mine]
>= MinLbs[elem];
s.t. TotalWeight: sum{mine in MINES} Tons[mine] = 1;
```

A Fixed Income Example

Investment Example

Bond	A	B	C	D	E
Yield	4.30%	2.70%	2.50%	2.20%	4.50%
Quality	2	2	1	1	5
Years to Maturity	9	15	4	3	2

Maximize Yield

Conditions:

	10	million to invest
at most	4	million total in C, D, and E
at most	1.4	average quality
at most	5	years average years to maturity

Challenge

- Build a linear model

A Fixed Income Example

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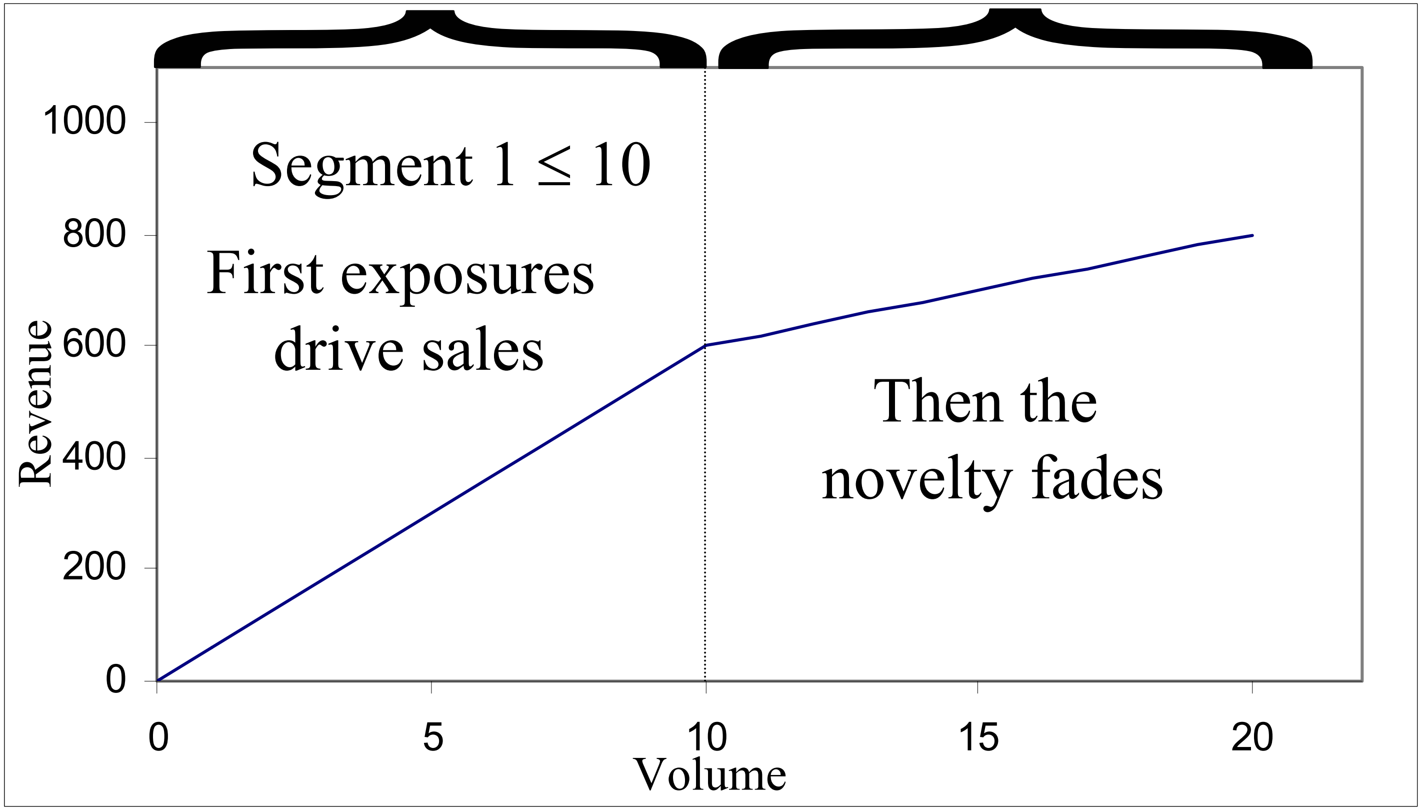
Diseconomies of Scale

- If we are minimizing cost
  - Unit cost increases with volume
- If we are maximizing profit
  - Unit profit decreases with volume
- Inherent incentive towards small volumes

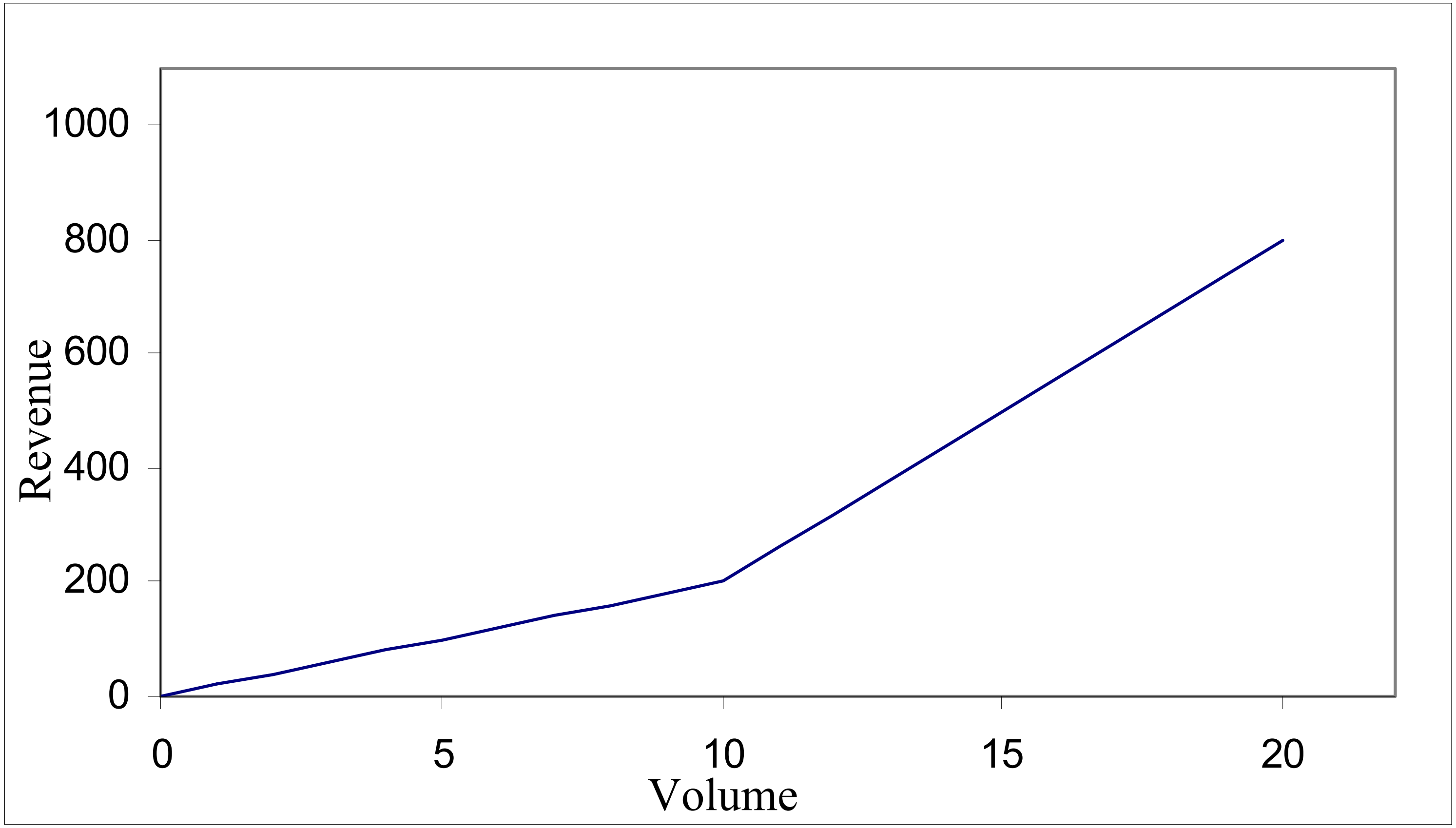
Example: Marketing

Total Volume = Segment 1 + Segment 2



Revenue = 60*Segment 1 + 20*Segment 2

What about....



A Financial Application

- Financial Analysis
  - Define goals
  - Assess Risk vs. Return
- Asset Allocation
  - Allocate assets among classes of investments
  - Track and forecast market swings
- Fund Selection
  - No-Load Funds
  - Established Performance

Asset Allocation

- Asset Classes
  - Treasury Bills
  - Small Value Funds
  - Large Growth Funds
  - Europe
  - Emerging Markets
  - HighYield
- Asset Allocation
  - Each Investor has target for each asset class

Fund Selection

- Screen Funds
- Rank in each Class for each Fund
  - Fidelity Equity Income II
    - LV LG SV JA EU GV HY
    - 66 4 16 2 1 6 5
    - Roughly speaking, the rank is the % of each funds investments that is in the asset class, e.g., Fidelity Equity Income II has 4% of its assets in Large Growth.
- Select Funds that meet the target allocation
  - Minimize the total “deviation” from the targets
  - Deviation is |Actual - Target|

Example Data

Fund Ratings

Fund Name	T-Bill	Large Value	Large Growth	Small Value	Small Growth	Japan	Pacific	Europe	Emerging Markets	Government	High Yield	International Bonds	Gold
Fidelity Adv Equity	7	71	2	6	7	2	0	0	5	0	0	0	0
Fidelity Advisor Gro	0	48	5	26	7	0	0	11	2	0	0	0	0
Fidelity Equity-Income	0	60	5	20	0	3	0	0	3	0	9	0	0
Fidelity Equity Income-II	0	66	4	16	0	2	1	0	6	0	5	0	0
Fidelity Growth/Income	2	47	0	17	11	3	0	5	2	0	12	0	0
Fidelity Ins Cash Po	100	0	0	0	0	0	0	0	0	0	0	0	0
Fidelity Investment	0	0	0	2	0	0	0	0	4	92	1	0	1
Fidelity Intermediat	13	0	0	0	0	0	0	0	0	83	0	3	0
Fidelity Limited Ter	5	18	0	0	0	0	4	0	0	45	28	0	0
Fidelity Mortgage Se	53	0	0	0	0	2	1	3	0	34	7	0	0
Fidelity Retirement	0	8	35	24	16	1	0	3	11	0	0	0	0
Fidelity Short-Term	44	0	0	0	0	0	0	0	6	25	23	3	0
Fidelity Value Fund	0	50	5	31	1	4	0	8	2	0	0	0	0
Fidelity Worldwide F	0	27	0	14	0	11	0	37	11	0	0	0	0
Targets	43	3	3	5	4	10	2	5	10	15	0	0	0

Example

- If we allocate 50% to the two funds…

Asset Classes	Fidelity Adv Equity	Fidelity Advisor Gro	Implied Allocation to Asset Classes	Target Allocation	Deviation
T-Bill	7	0	3.5	43	39.5
Large Value	71	48	59.5	3	56.5
Large Growth	2	5	3.5	3	0.5
Small Value	6	26	16	5	11
Small Growth	7	7	7	4	3
Japan	2	0	1	10	9
Pacific	0	0	0	2	2
Europe	0	11	5.5	5	0.5
Emerging Markets	5	2	3.5	10	6.5
Government	0	0	0	15	15
High Yield	0	0	0	0	0
International Bonds	0	0	0	0	0
Gold	0	0	0	0	0
	50%	50%		**Total**	**143.5**

Challenge #2

- Build a **<u>linear</u>** model to find a best portfolio.
- First build your model in Excel (Use the file Portfolio.xls)
- Then build your model in AMPL (Use the file Portfolio.mdb)
- Deliverables
  - Models (Self documenting)
  - Solutions (Self documenting)
- Due: Beginning of Lecture #7