Class 5 Outline

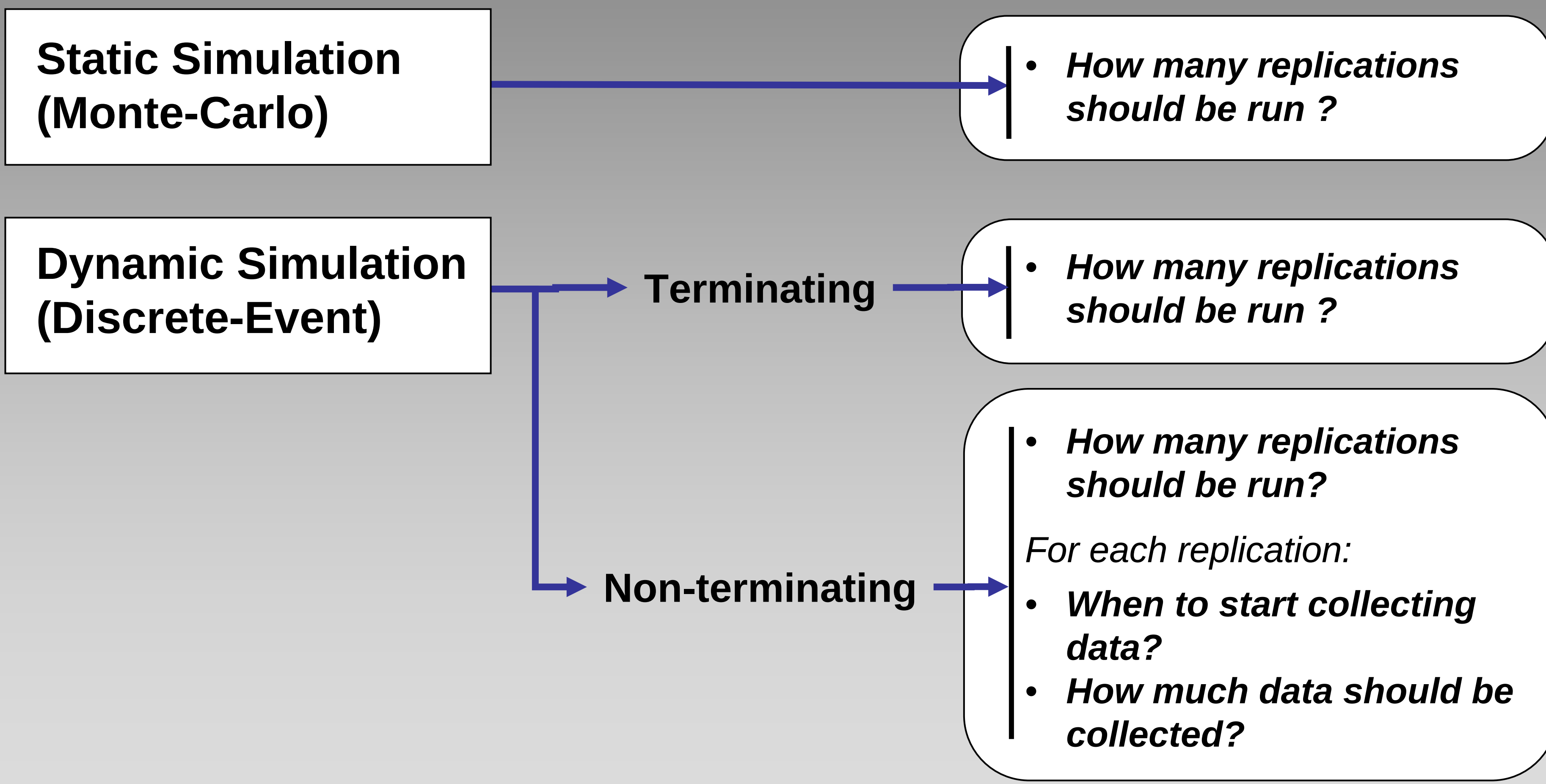
Experiment Design and Output Analysis:

1. **Theory and Construction of Confidence Intervals**

2. **Monte-Carlo and Terminating Simulations**

3. **Non-Terminating Simulations (Steady State)**

Experimental Design Issues



How Many Replications?

- **Simulation output: $Y_1, Y_2, Y_3, \ldots, Y_n$**
  We (typically) want to estimate E[Y]!

1. **What is the accuracy of the estimator:**

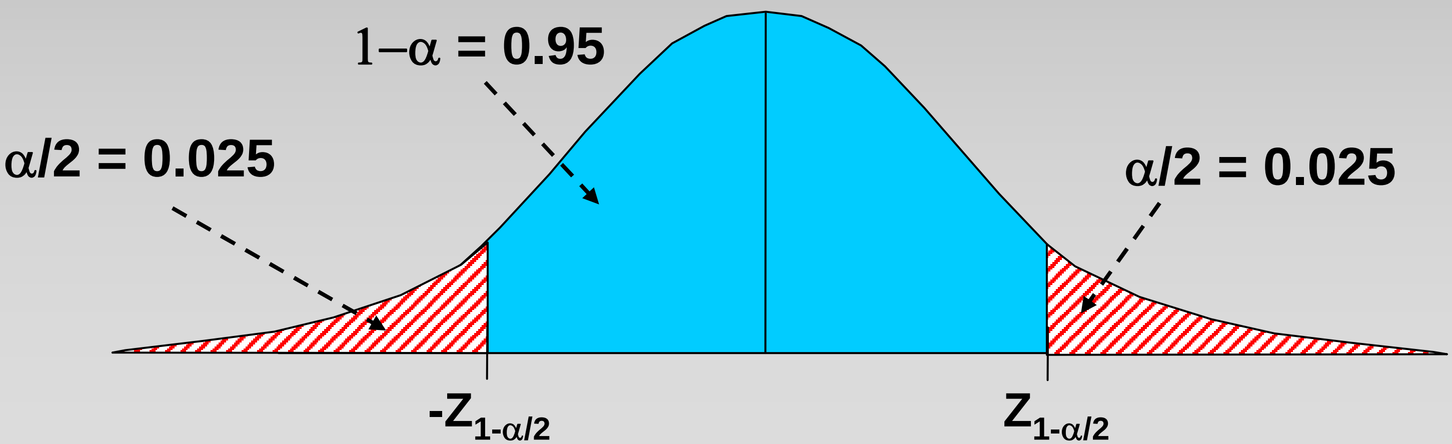
$$Y(n) = (Y_1+Y_2+Y_3+ \ldots+Y_n) / n \ ?$$

2. **How much should n be (number of independent replications) in order to guarantee a given estimation accuracy?**

What is a “Fractile”?

- **Let α be a number in (0,1) e.g. 0.05**
- **$z_{1-\alpha/2}$ is the (1-α/2)-th fractile of a standard normal distribution, and is defined as:**

$$P \;\; N \;\; 0,1 \;\; \le z_{1- \;\; /2} \qquad 1 - \;\; /2$$



Statistical Estimation Theory

- **Define:**

$$Y\ n \qquad \frac{1}{n}\sum_{i\ 1}^{n} Y_i \qquad\qquad S^2\ n \qquad \frac{1}{n-1}\sum_{i\ 1}^{n} Y_i - Y\ n \quad ^2$$

- **A version of the CLT says that when n → ∞ :**

$$\frac{Y\ n\ - E\ Y}{\sqrt{\frac{S^2\ n}{n}}} \rightarrow N\ 0,1$$

- **For smaller (n<30) values of n, a good approx. is:**

$$\frac{Y\ n\ - E\ Y}{\sqrt{\frac{S^2\ n}{n}}} \rightarrow t_{n-1}$$

So What?

- **From the CLT:**

$$\frac{Y \; n \; - E \; Y}{\sqrt{\frac{S^2 \; n}{n}}} \rightarrow N \; 0, 1$$

- **So that when n is large, we can write:**

$$P \; -z_{1- \;\; /2} \leq \frac{Y \; n \; - E \; Y}{\sqrt{\frac{S^2 \; n}{n}}} \leq z_{1- \;\; /2} \qquad 1 -$$

- **Re-arranging gives:**

$$P \; Y \; n \; - z_{1- \;\; /2} \sqrt{\frac{S^2 \; n}{n}} \leq E \; Y \; \leq Y \; n \qquad z_{1- \;\; /2} \sqrt{\frac{S^2 \; n}{n}} \qquad 1 -$$

- **This is the definition of a (1-α)% confidence interval!!!**

Building Confidence Intervals

- **For n large (n>30), the (1- α)% confidence interval is:**

$$Y\ n \qquad z_{1-\ /2}\sqrt{\frac{S^2\ n}{n}}$$

fractile of the std. normal distribution

- **For n small (n<30), use:**

$$Y\ n \qquad t_{n-1,1-\ /2}\sqrt{\frac{S^2\ n}{n}}$$

fractile of the t (student) distribution with n-1 d.f.

So, How Many Replications?

- W = UB – LB is the width of the confidence interval, centered around Y(n)
- A measure of the relative estimation error is thus **W / Y(n)**
- So a good termination criteria is to set an estimation error β and run a number of replications **n** such that:

$$W / Y(n) < \beta$$

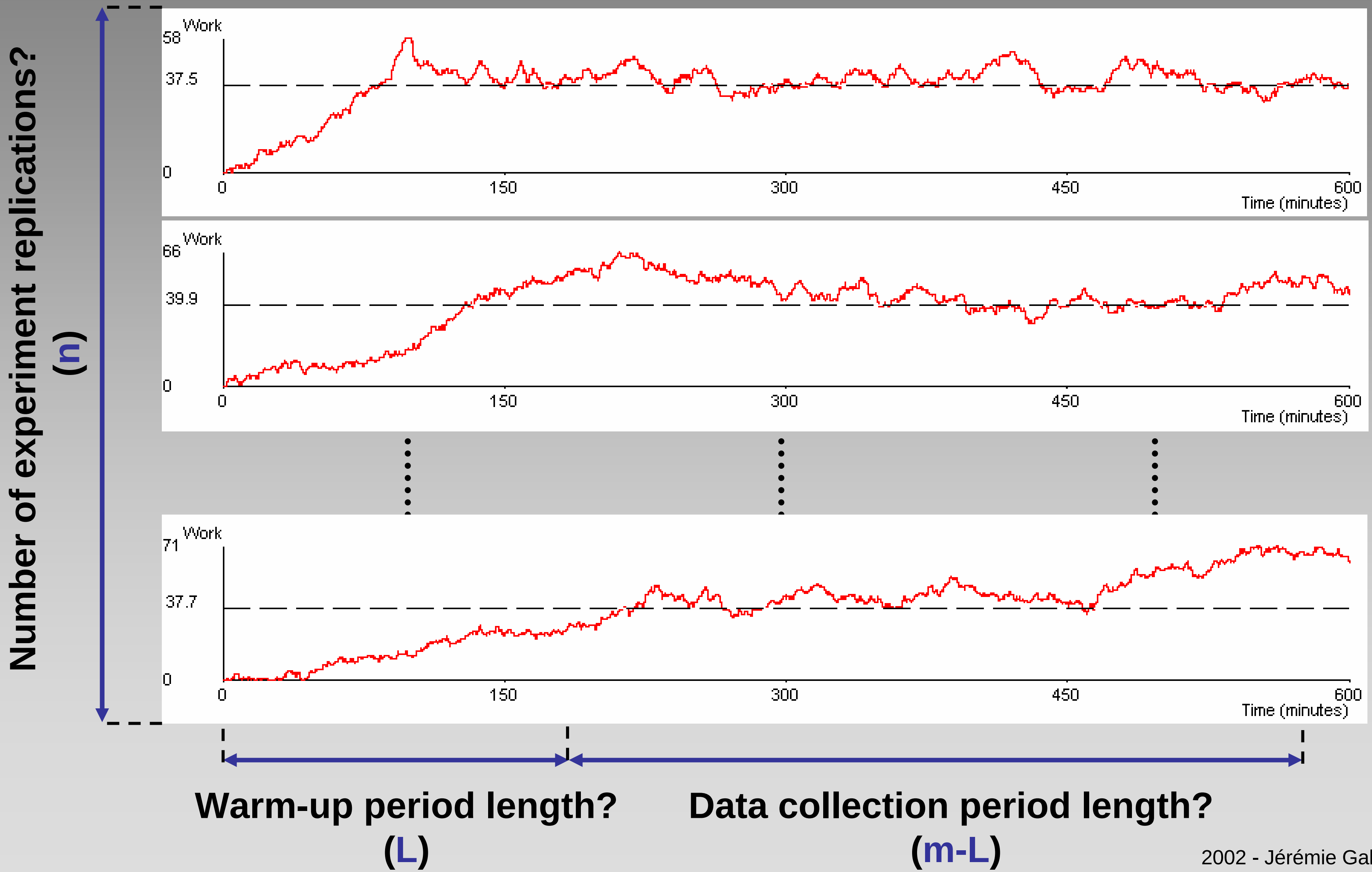
Example

Suppose we build a simulation model to assess the weekly cost of a proposed supply chain inventory policy

From the simulation data output (Spreadsheet "Confidence Interval" on SloanSpace), we want to estimate:

1. **The average weekly cost C with CI**
2. **P(C>$2M) with CI**

Steady-State Analysis



Steady-State Analysis

- **Let $Y_1, Y_2, Y_3, \ldots, Y_t, \ldots$ be a stochastic process (e.g. queue length at time index t).**

  We want to estimate lim E[Y_t] when $t \to \infty$!

- **Data Y_{ij} : j-th observation in the i-th replication**

1st replication → $Y_{11}, Y_{12}, Y_{13}, \ldots, Y_{1j}, \ldots, Y_{1m}$

2nd replication → $Y_{21}, Y_{22}, Y_{23}, \ldots, Y_{2j}, \ldots, Y_{2m}$

⋮

last replication → $Y_{n1}, Y_{n2}, Y_{n3}, \ldots, Y_{nj}, \ldots, Y_{nm}$

first observation ………………… last observation

Replication/Deletion Approach

(Y_{ij} : j-th observation in the i-th replication)

- **For each replication i, instead of the estimator:**

  $$Y_i(m) = (Y_{i1}+Y_{i2}+Y_{i3}+ \ldots +Y_{im}) / m,$$

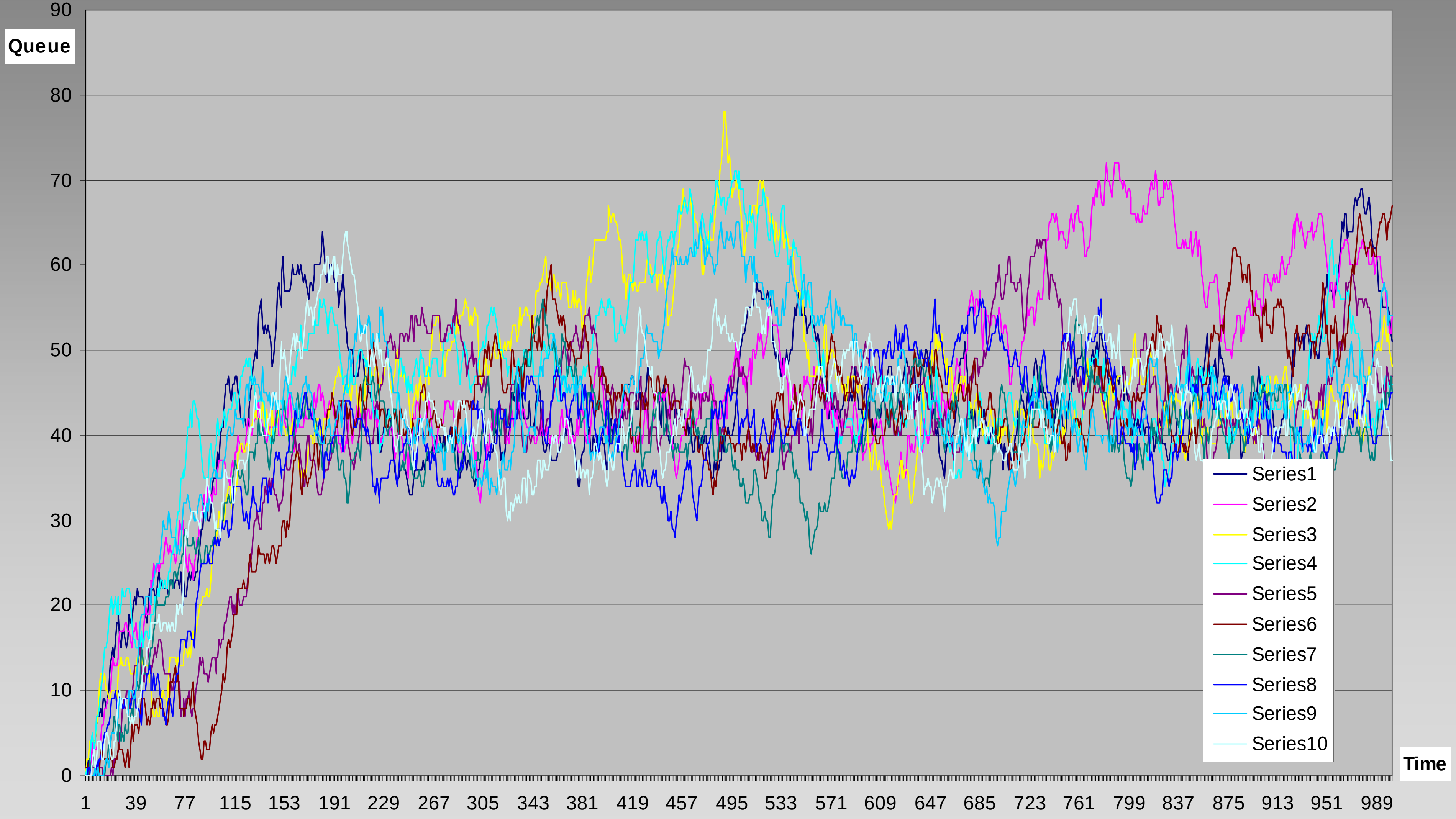
  consider the modified estimator:

  $$Y_i(m,L) = (Y_{iL}+Y_{i(L+1)}+Y_{i(L+2)}+ \ldots +Y_{im}) / (m-L+1)$$

- **An estimator for lim $E[Y_t]$ when $t \to \infty$ is then**

  $$Y(n) = (Y_1(m,L) + Y_2(m,L) \ldots + Y_n(m,L)) / n$$

Experimental Data Plot



2002 - Jérémie Gallien

Welch's Method for Warm-up

(Y_{ij} : j-th observation in the i-th replication)

- **Compute the average across replications for each time point:**

$$Y_t[n] = (Y_{1t}+Y_{2t}+Y_{3t}+ \ldots +Y_{nt}) / n$$

- **Welch's method is to plot the moving average process of $Y_t[n]$ based on various lags w:**

$$Y_t[n,w] = (Y_{t-w}[n] + \ldots + Y_t[n] + \ldots +Y_{t+w}[n]) / (2w + 1)$$

Welch’s Method for Warm-up