MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.265/15.070 Fall 2013
Problem Set 3 due 10/23/2013

Problem 1. Let B be standard Brownian motion. Show that $\mathbb{P}(\limsup_{t\to\infty} B(t) = \infty) = 1$.

Problem 2. (a) Consider the following sequence of partitions $\Pi_n, n = 1, 2, \ldots$ of $[0, T]$ given by $t_i = \frac{i}{n}, 0 \le i \le n$. Prove that quadratic variation of a standard Brownian motion almost surely converges to T: $\lim_n Q(\Pi_n, B) = 1$ a.s., even though $\sum_n \Delta(\Pi_n) = \sum_n 1/n = \infty$.

(b) Suppose now the partition is generated by drawing n independent random values $t_k = U_k, 1 \le k \le n$ drawn uniformly from $[0, T]$ and independently from the Brownian motion. Prove that $\lim_n Q(\Pi_n, B) = T$ a.s. Note, almost sure is with respect to the probability space of both the Brownian motion probability and uniform sampling.

Problem 3. Suppose $X \in \mathcal{F}$ is independent from $\mathcal{G} \subset \mathcal{F}$. Namely, for every measurable $A \subset \mathbb{R}, B \in \mathcal{G}$ $\mathbb{P}(\{X \in A\} \cap B) = \mathbb{P}(X \in A)\mathbb{P}(B)$. Prove that $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

Problem 4. Consider an assymetric simple random walk $Q(t)$ on $\mathbb{Z}$ given by $\mathbb{P}(Q(t+1) = x+1|Q(t) = x) = p$ and $\mathbb{P}(Q(t+1) = x-1|Q(t) = x) = 1-p$ for some $0 < p < 1$.

1. Construct a function of the state $\phi(x), x \in \mathbb{Z}$ such that $\phi(Q(t))$ is a martingale.
2. Suppose $Q(0) = z > 0$ and $p > 1/2$. Compute the probability that the random walk never hits 0 in terms of z, p.

Problem 5. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider a sequence of random variables $X_1, X_2, \ldots, X_n$ and σ-fields $\mathcal{F}_1, \ldots, \mathcal{F}_n \subset \mathcal{F}$ such that $\mathbb{E}[X_j|\mathcal{F}_{j-1}] = X_{j-1}$ and $\mathbb{E}[X_j^2] < \infty$.

1. Prove directly (without using Jensen's inequality) that $\mathbb{E}[X_j^2] \ge \mathbb{E}[X_{j-1}^2]$ for all $j = 2, \ldots, n$. *Hint*: consider $(X_j - X_{j-1})^2$.

2. Suppose $X_n = X_1$ almost surely. Prove that in this case $X_1 = \ldots = X_n$ almost surely.

Problem 6. The purpose of this exercise is to extend some of the stopping times theory to processes which are (semi)-continuous. Suppose X_t is a continuous time submartingale adopted to $\mathcal{F}_t, t \in \mathbb{R}_+$ and T is a stopping time taking values in $\mathbb{R} \cup \{\infty\}$. Suppose additionally that X_t is a.s. a right-continuous function with left limits (RCLL).

(a) Suppose there exists a countably infinite strictly increasing sequence $t_n \in \mathbb{R}_+, n \geq 0$, such that $\mathbb{P}(T \in \{t_n, n \geq 0\} \cup \{\infty\}) = 1$. Emulate the proof of the discrete time processes to show that $X_{t \wedge T}, t \in \mathbb{R}_+$ is a submartingale.

(b) Given a general stopping time T taking values in $\mathbb{R}_+ \cup \{\infty\}$, consider a sequence of r.v. T_n defined by $T_n(\omega) = \frac{k}{2^n}, k = 1, 2, \ldots$ if $T(\omega) \in [\frac{k-1}{2^n}, \frac{k}{2^n})$ and $T_n(\omega) = \infty$ if $T(\omega) = \infty$. Establish that T_n is a stopping time for every n.

(c) Suppose the submartingale X_t is in $\mathbb{L}_2$, namely $\mathbb{E}[X_t^2] < \infty, \forall t$. Show that $X_{T \wedge t}$ is a submartingale as well.

Hint: Use part (b), Doob-Kolmogorov inequality and the Dominated Convergence Theorem.

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