15.093 Optimization Methods

Lecture 3: The Simplex Method

1 Outline

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- Reduced Costs
- Optimality conditions
- Improving the cost
- Unboundness
- The Simplex algorithm
- The Simplex algorithm on degenerate problems

2 Matrix View

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$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

$\boldsymbol{x} = (\boldsymbol{x}_B, \boldsymbol{x}_N)$ $\quad$ $\boldsymbol{x}_B$ basic variables
$\boldsymbol{x}_N$ non-basic variables

$\boldsymbol{A} = [\boldsymbol{B}, \boldsymbol{N}]$
$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Rightarrow \quad \boldsymbol{B} \cdot \boldsymbol{x}_B + \boldsymbol{N} \cdot \boldsymbol{x}_N = \boldsymbol{b}$
$\Rightarrow \boldsymbol{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b}$
$\Rightarrow \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N$

2.1 Reduced Costs

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$$\begin{aligned} z &= \boldsymbol{c}_B'\boldsymbol{x}_B + \boldsymbol{c}_N'\boldsymbol{x}_N \\ &= \boldsymbol{c}_B'(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N\boldsymbol{x}_N) + \boldsymbol{c}_N'\boldsymbol{x}_N \\ &= \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}_N' - \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{N})\boldsymbol{x}_N \end{aligned}$$

$$\boxed{\overline{c}_j = c_j - \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{A}_j \quad \text{reduced cost}}$$

2.2 Optimality Conditions

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Theorem:

- $\boldsymbol{x}$ BFS associated with basis B
- $\overline{\boldsymbol{c}}$ reduced costs
  Then
- If $\overline{\boldsymbol{c}} \geq \boldsymbol{0} \Rightarrow \boldsymbol{x}$ optimal
- $\boldsymbol{x}$ optimal and non-degenerate $\Rightarrow \overline{\boldsymbol{c}} \geq \boldsymbol{0}$

2.3 Proof

- $\boldsymbol{y}$ arbitrary feasible solution
- $\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x} \Rightarrow \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} \Rightarrow \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$

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$$\Rightarrow \boldsymbol{B}\boldsymbol{d}_B + \sum_{i\in N} \boldsymbol{A}_i d_i = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{d}_B = -\sum_{i\in N} \boldsymbol{B}^{-1}\boldsymbol{A}_i d_i$$

$$\begin{aligned} \Rightarrow \boldsymbol{c}'\boldsymbol{d} &= \boldsymbol{c}'_B\boldsymbol{d}_B + \sum_{i\in N} c_i d_i \\ &= \sum_{i\in N}(c_i - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{A}_i)d_i = \sum_{i\in N} \overline{c}_i d_i \end{aligned}$$

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- Since $y \geq 0$ and $x_i = 0, i \in N$, then $d_i = y_i - x_i \geq 0, i \in N$
- $\boldsymbol{c}'\boldsymbol{d} = \boldsymbol{c}'(\boldsymbol{y} - \boldsymbol{x}) \geq 0 \ \Rightarrow \boldsymbol{c}'\boldsymbol{y} \geq \boldsymbol{c}'\boldsymbol{x}$

  $\Rightarrow \boldsymbol{x}$ optimal

  (b) in BT, Theorem 3.1

3 Improving the Cost

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- Suppose $\overline{c}_j = c_j - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{A}_j < 0$

  Can we improve the cost?
- Let $\boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j$

  $d_j = 1,\ d_i = 0,\ i \neq B(1), \ldots, B(m), j.$
- Let $\boldsymbol{y} = \boldsymbol{x} + \theta \cdot \boldsymbol{d}, \quad \theta > 0$ scalar

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$$\begin{aligned} \boldsymbol{c}'\boldsymbol{y} - \boldsymbol{c}'\boldsymbol{x} &= \theta \cdot \boldsymbol{c}'\boldsymbol{d} \\ &= \theta \cdot (\boldsymbol{c}'_B\boldsymbol{d}_B + c_j d_j) \\ &= \theta \cdot (c_j - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{A}_j) \\ &= \theta \cdot \overline{c}_j \end{aligned}$$

Thus, if $\overline{c}_j < 0$ cost will decrease.

4 Unboundness

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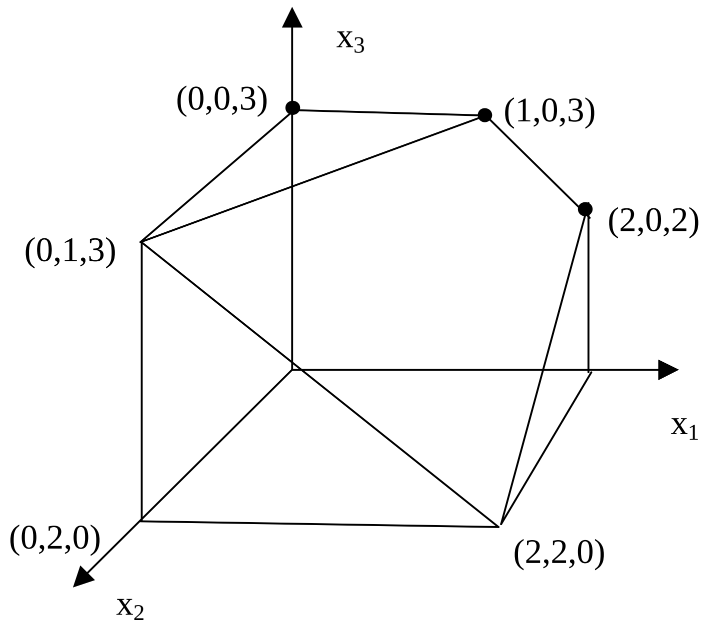
- Is $\boldsymbol{y} = \boldsymbol{x} + \theta \cdot \boldsymbol{d}$ feasible?

  Since $\boldsymbol{A}\boldsymbol{d} = 0 \Rightarrow \boldsymbol{A}\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
- $\boldsymbol{y} \geq \boldsymbol{0}$?

  If $\boldsymbol{d} \geq \boldsymbol{0} \Rightarrow \boldsymbol{x} + \theta \cdot \boldsymbol{d} \geq \boldsymbol{0} \quad \forall\, \theta \geq 0$

  $\Rightarrow$ objective unbounded.



5 Improvement

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If $d_i < 0$, then

$$x_i + \theta d_i \geq 0 \Rightarrow \theta \leq -\frac{x_i}{d_i}$$

$\Rightarrow \theta^* = \min_{\{i|d_i<0\}} \left(-\frac{x_i}{d_i}\right)$

$\Rightarrow \theta^* = \min_{\{i=1,\ldots,m|d_{B(i)}<0\}} \left(-\frac{x_{B(i)}}{d_{B(i)}}\right)$

5.1 Example

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$$\begin{array}{lllll} \min & x_1+ & 5x_2 & -2x_3 & \\ \text{s.t.} & x_1+ & x_2+ & x_3 & \leq 4 \\ & x_1 & & & \leq 2 \\ & & & x_3 & \leq 3 \\ & & 3x_2+ & x_3 & \leq 6 \\ & x_1, & x_2, & x_3 & \geq 0 \end{array}$$

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$$\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \boldsymbol{A}_4 & \boldsymbol{A}_5 & \boldsymbol{A}_6 & \boldsymbol{A}_7 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

B = $[\boldsymbol{A}_1, \boldsymbol{A}_3, \boldsymbol{A}_6, \boldsymbol{A}_7]$
BFS: $\boldsymbol{x} = (2, 0, 2, 0, 0, 1, 4)'$

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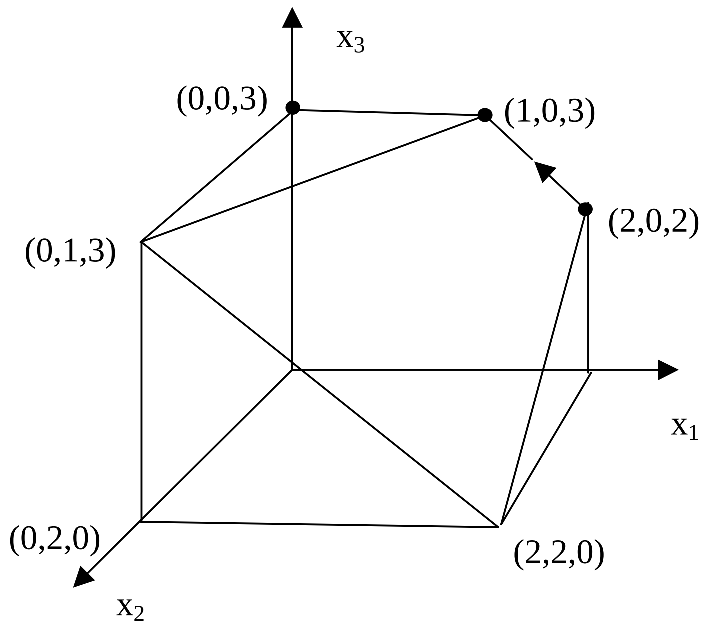

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \overline{\boldsymbol{c}}' = (0,7,0,2,-3,0,0)$$

$$d_5 = 1, d_2 = d_4 = 0, \quad \begin{pmatrix} d_1 \\ d_3 \\ d_6 \\ d_7 \end{pmatrix} = -\boldsymbol{B}^{-1}\boldsymbol{A}_5 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

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$$\boldsymbol{y}' = \boldsymbol{x}' + \theta\boldsymbol{d}' = (2-\theta, 0, 2+\theta, 0, \theta, 1-\theta, 4-\theta)$$

What happens as θ increases?

$\theta^* = \min_{\{i=1,\ldots,m|d_{B(i)}<0\}} \left(-\frac{x_{B(i)}}{d_i}\right) =$

$\min\left(-\frac{2}{(-1)}, -\frac{1}{(-1)}, -\frac{4}{(-1)}\right) = 1.$

$l = 6$ ($\boldsymbol{A}_6$ exits the basis).

New solution

$\boldsymbol{y} = (1,0,3,0,1,0,3)'$

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New basis $\overline{\boldsymbol{B}} = (\boldsymbol{A}_1, \boldsymbol{A}_3, \boldsymbol{A}_5, \boldsymbol{A}_7)$

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$$\overline{\boldsymbol{B}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \overline{B}^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$\overline{\boldsymbol{c}}' = \boldsymbol{c}' - \boldsymbol{c}'_{\overline{B}}\overline{\boldsymbol{B}}^{-1}\boldsymbol{A} = (0,4,0,-1,0,3,0)$

Need to continue, column $\boldsymbol{A}_4$ enters the basis.

6 Correctness

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$$-\frac{x_{B(l)}}{d_{B(l)}} = \min_{i=1,\ldots,m,d_{B(i)}<0} \left(-\frac{x_{B(i)}}{d_{B(i)}}\right) = \theta^*$$

Theorem

- $\overline{\boldsymbol{B}} = \{\boldsymbol{A}_{B_{(i)},i\neq l}, \boldsymbol{A}_j\}$ basis
- $\boldsymbol{y} = \boldsymbol{x} + \theta^* \boldsymbol{d}$ is a BFS associated with basis $\overline{\boldsymbol{B}}$.

7 The Simplex Algorithm

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1. Start with basis $\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}]$
   and a BFS $\boldsymbol{x}$.
2. Compute $\overline{c}_j = c_j - \boldsymbol{c}'_B \boldsymbol{B}^{-1} \boldsymbol{A}_j$
   - If $\overline{c}_j \geq 0$; x optimal; stop.
   - Else select $j : \overline{c}_j < 0$.

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3. Compute $\boldsymbol{u} = -\boldsymbol{d} = \boldsymbol{B}^{-1} \boldsymbol{A}_j$.
   - If $\boldsymbol{u} \leq \boldsymbol{0} \Rightarrow$ cost unbounded; stop
   - Else
4. $\theta^* = \min_{1\leq i\leq m, u_i>0} \frac{x_{B(i)}}{u_i} = \frac{u_{B(l)}}{u_l}$
5. Form a new basis by replacing $\boldsymbol{A}_{B(l)}$ with $\boldsymbol{A}_j$.
6. $y_j = \theta^*$
   $y_{B(i)} = x_{B(i)} - \theta^* u_i$

7.1 Finite Convergence

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Theorem:

- $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}\} \neq \emptyset$
- Every BFS non-degenerate
  Then
- Simplex method terminates after a finite number of iterations
- At termination, we have optimal basis $\boldsymbol{B}$ or we have a direction $\boldsymbol{d} : \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}, \boldsymbol{d} \geq \boldsymbol{0}, \boldsymbol{c}'\boldsymbol{d} < \boldsymbol{0}$ and optimal cost is $-\infty$.

7.2 Degenerate problems

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- θ^* can equal zero (why?) $\Rightarrow \boldsymbol{y} = \boldsymbol{x}$, although $\overline{\boldsymbol{B}} \neq \boldsymbol{B}$.
- Even if $\theta^* > 0$, there might be a tie

$$\min_{1 \le i \le m, u_i > 0} \frac{x_{B(i)}}{u_i} \Rightarrow$$

  next BFS degenerate.
- Finite termination not guaranteed; cycling is possible.

7.3 Avoiding Cycling

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- Cycling can be avoided by carefully selecting which variables enter and exit the basis.
- Example: among all variables $\overline{c}_j < 0$, pick the smallest subscript; among all variables eligible to exit the basis, pick the one with the smallest subscript.

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