Telecommunication System Design: Minimum-Cost Embeddings of Reliable Virtual Private Networks

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Overview

- The Project's Origin
- The Problem's Origin
- An Integer Programming Model
- A Lagrangian Relaxation
- A Comparison of Lower Bounds
- A Branch&Bound Approach
- Computational Results
- Extensions

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The Project's Origin

- Symposium on Operations Research 1994, Berlin.
- R&D Division, Deutsche Telekom AG, Darmstadt.
- Math. Dept., Berlin University of Technology:
  - Ewgenij Gawrilow (Programmer),
  - Olaf Jahn (Research Assistant),
  - Rolf H. Möhring (Principal Investigator),
  - Martin Oellrich (Research Assistant),
  - Andreas S. Schulz (Principal Investigator).
- Official Start: July 1, 1995. 1 year.

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The Problem's Origin

Market Liberalization

Prior to Deregulation.

Operation + management of network infrastructure **and** provision of network services organized as integrated process.

Post Deregulation.

Competing providers of network services lease required transport capacity from carriers of physical transmission networks.

Resulting subnetworks are independently operated.

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The Problem's Origin

A VPN appears to be exclusively controlled and managed by a customer alone.

In reality, it consists of a number of lines leased from a carrier.

VPNs are typically deployed as data service networks, or backbones for mobile and ATM networks.

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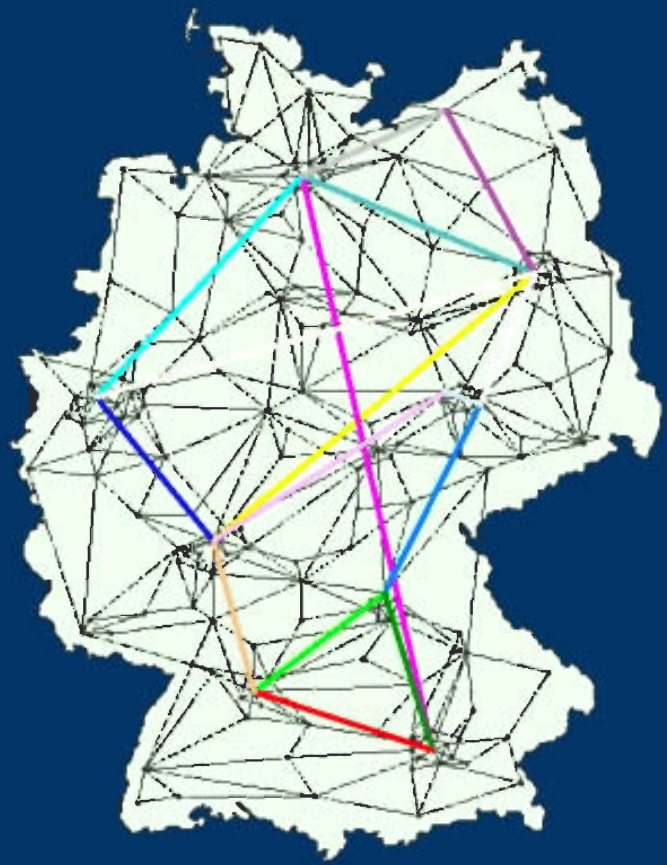
The Problem's Origin

Virtual Private Networks



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The Problem's Origin

Carrier's Objective

The carrier has to balance two conflicting goals:

- On the one hand, customers require reliability.
- On the other hand, costs have to be kept at a minimum in order to be competitive.

Note. In network engineering, reliability is a key issue. Keeping their networks stable and operable are primary goals of all service providers.

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The Problem's Origin

Disjoint Routing

A VPN is **reliable** when no single fault in one physical trunk can affect more than one of the logical connections in the VPN. One can accommodate this request by routing no two connections over a common trunk.

Technically speaking, the different leased lines must be routed **disjointly**.

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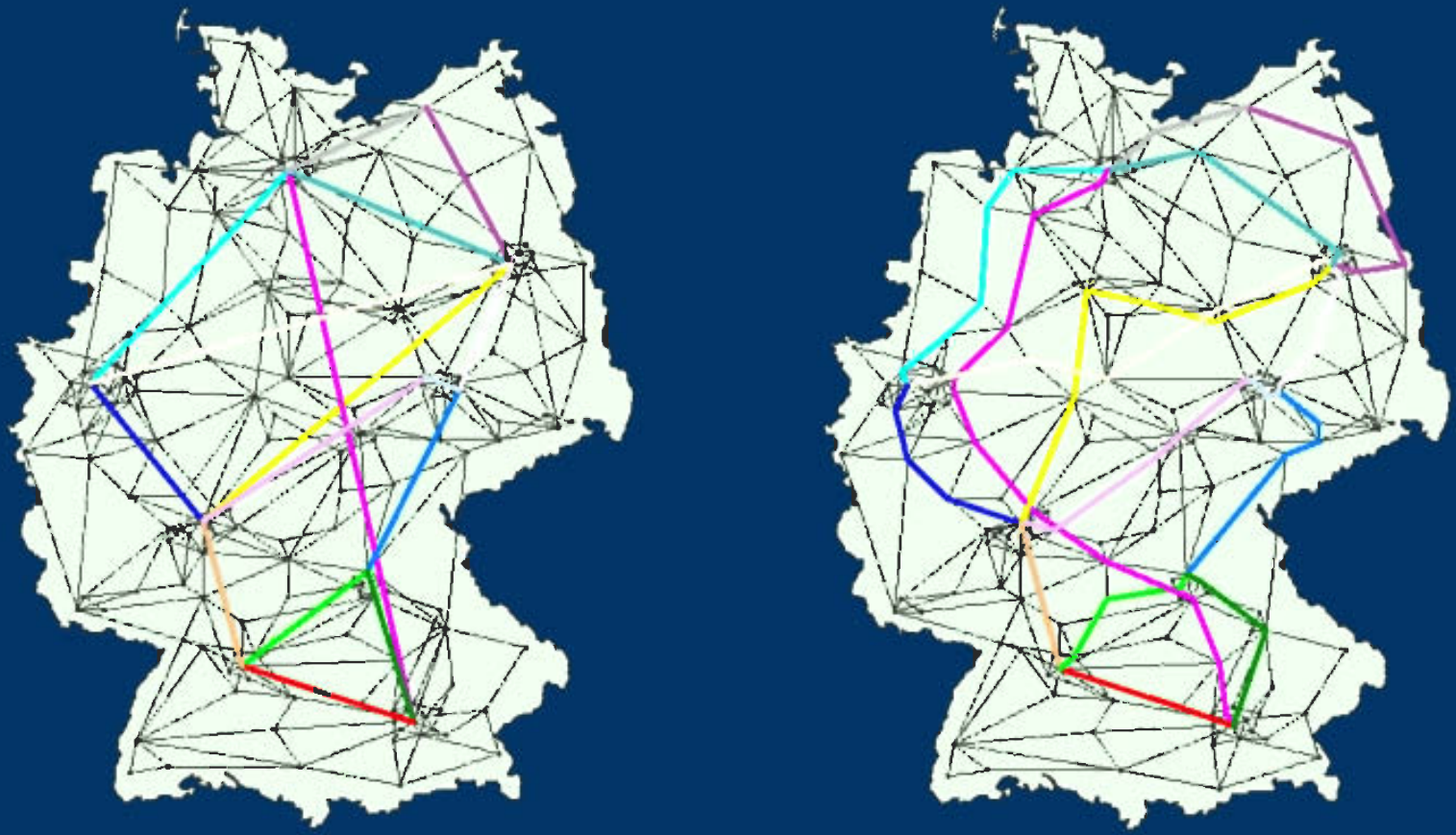
The Problem

Disjoint Routing



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Mathematical Model

Disjoint Paths Problem

THE MINIMUM COST DISJOINT PATHS PROBLEM.

Input: A graph $G = (V, E)$,
$c : E \to \mathbb{N}$,
k pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ of terminals.

Goal: Find k edge-disjoint paths $P_1, \ldots, P_k$
in G connecting $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$
such that $\sum_{i=1}^{k} c(P_i)$ is minimum.

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Computational Complexity

- How difficult is the MINIMUM COST DISJOINT PATHS PROBLEM?
- What if we disregard the disjointness constraint?

???

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Mathematical Model

Multicommodity Flow

$$\min \sum_{\ell=1}^{k} \sum_{(i,j)\in A} c_{ij}\, x_{ij}^{\ell}$$

$$\text{s.t.} \sum_{j} x_{ij}^{\ell} - \sum_{j} x_{ji}^{\ell} = b_i^{\ell} \qquad \forall i \in V, \forall \ell \in \{1, \ldots, k\},$$

$$\sum_{\ell=1}^{k} (x_{ij}^{\ell} + x_{ji}^{\ell}) \leq 1 \qquad \forall \{i,j\} \in E,$$

$$x_{ij}^{\ell} \in \{0,1\} \qquad \forall (i,j) \in A, \forall \ell \in \{1, \ldots, k\}.$$

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Mathematical Model

Multicommodity Flow

Notation

Directed Network:

$$A := \{\, (u,v),\ (v,u) \mid \{u,v\} \in E \,\}$$

Supplies/Demands:

$$b_i^\ell := \begin{cases} 1, & \text{if } i = s_\ell, \\ -1, & \text{if } i = t_\ell, \\ 0, & \text{otherwise.} \end{cases}$$

Decisions:

$$x_{ij}^\ell := \begin{cases} 1, & \text{if } \{i,j\} \text{ belongs to path } P_\ell, \\ 0, & \text{otherwise.} \end{cases}$$

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Lower Bounds

Importance

- Proving optimality.
- Identifying near-optimal solutions.
- Reducing search space in enumerative approaches.

Your Ideas for Lower Bounds?

1. Combinatorial (Cheapest Paths)

2. Linear Programming Relaxation

3. Lagrangian Relaxation

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Lagrangian Relaxation

Dualizing "Bad" Constraints

$$\begin{aligned} \mathbf{OP}: \min_{x \in \{0,1\}} \quad & cx \\ \text{s.t.} \quad & Nx = b \\ & Ax \leq 1 \end{aligned}$$

We know that optimization over the constraints "$Nx = b,\ x \in \{0,1\}$" is easy.

The addition of the constraints "$Ax \leq 1$" makes the problem much more difficult.

Let $P = \{x \in \{0,1\} : Nx = b\}$.

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Lagrangian Relaxation

Dualizing "Bad" Constraints

$$\begin{aligned} \text{OP}: \min_x \; & c\,x \\ \text{s.t.} \; & Ax \leq 1 \\ & x \in P \end{aligned}$$

The Lagrangian is:

$$L(x,u) = c\,x + u\,(Ax - 1) = -u\,1 + (c + u\,A)x$$

And the Lagrangian dual, for $u \geq 0$:

$$\begin{aligned} L^*(u) := \min_x \; & -u\,1 + (c + u\,A)x \\ \text{s.t.} \; & x \in P \end{aligned}$$

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Lagrangian Relaxation

Intuitive Interpretation

$$\min \sum_{\ell=1}^{k} \sum_{(i,j)\in A} c_{ij}\, x_{ij}^{\ell} + \sum_{\{i,j\}\in E} u_{ij} \Big(\sum_{\ell=1}^{k} (x_{ij}^{\ell} + x_{ji}^{\ell}) - 1 \Big)$$

$$\sum_{j} (x_{ij}^{\ell} - x_{ji}^{\ell}) = b_i^{\ell} \qquad \forall i, \forall \ell$$

$$x_{ij}^{\ell} \in \{0, 1\} \qquad \forall (i,j), \forall \ell$$

Repeatedly used arcs have higher (virtual) cost.

Unused arcs might become more attractive.

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Solving the Dual

$$L^*(u) := \begin{array}{ll} \min_x & -u\,1 + (c + u\,A)x \\ \text{s.t.} & x \in P \end{array}$$

$$\mathbf{D}: \max_u L^*(u)$$
$$\text{s.t.} \quad u \geq 0$$

Notice that $L^*(u)$ is easy to evaluate for any value of u, and so we attempt to get good lower bounds for **OP** by designing an algorithm to solve the dual problem **D**.

Which algorithm could we choose?

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Properties of the Dual

- The dual is a concave maximization problem.
- The dual function $L^*(u)$ is piece-wise linear.

$$L^*(u) = \min_{x \in P} L(x, u)$$

$$= \min\{-u\,1 + (c + u\,A)x^t \;:\; t = 1, \ldots, T\}$$

We may use a subgradient method to solve the dual.

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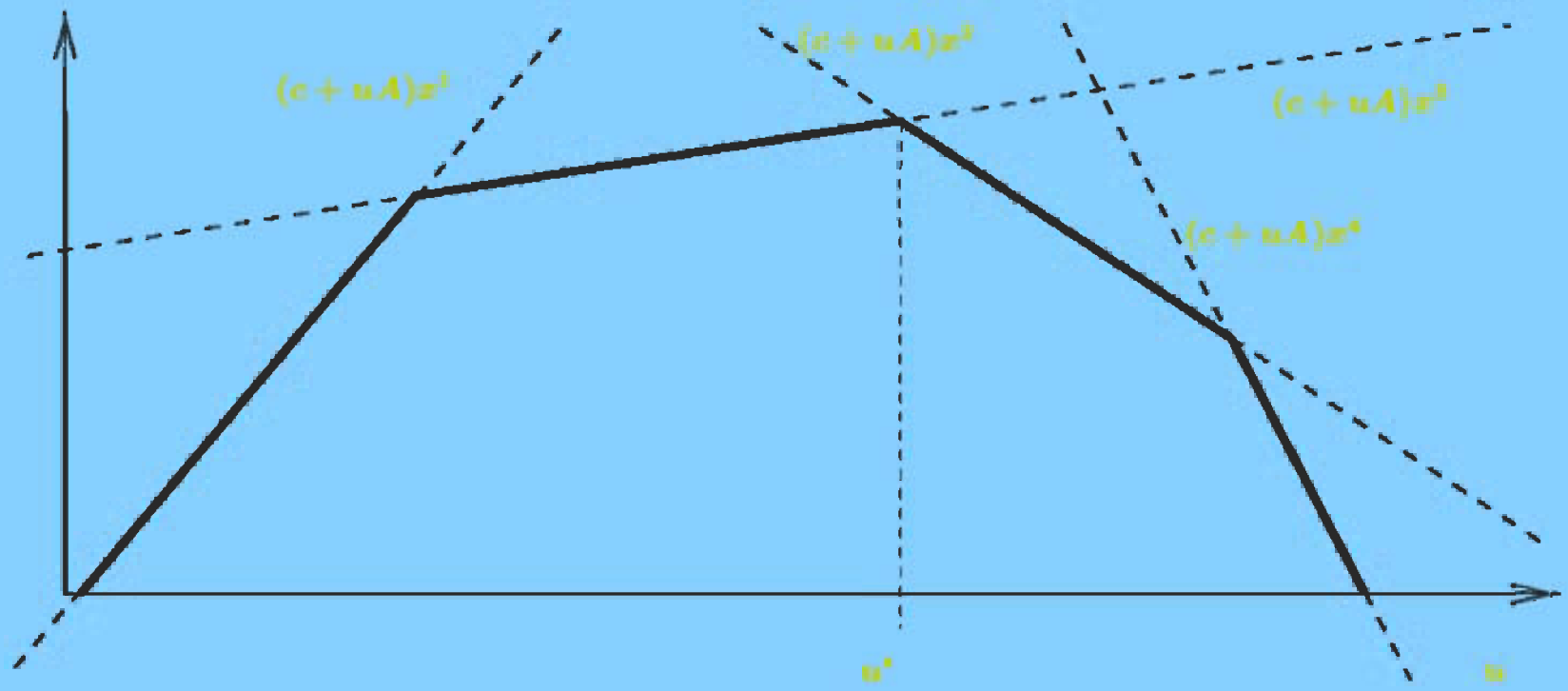
Lagrangian Relaxation

Properties of the Dual



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Combinatorial Subgradients

Theorem 1

Let x^* be an optimal solution to $\min_{x \in P} L(x, u)$, for some $u \geq 0$. Then, $A x^* - 1$ is a subgradient of $L^*(\,.\,)$ in u.

Remember d is a subgradient of L^* in u iff $L^*(v) - L^*(u) \leq d\,(v - u)$ for all v.

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Combinatorial Subgradients

Proof

$$
\begin{aligned}
L^*(v) - L^*(u) &= \min_{x \in P} L(x, v) - \min_{x \in P} L(x, u) \\
&= \min_{x \in P} L(x, v) - L(x^*, u) \\
&\leq L(x^*, v) - L(x^*, u) \\
&= (c\,x^* + v\,(A\,x^* - 1)) - (c\,x^* + u\,(A\,x^* - 1)) \\
&= (A\,x^* - 1)\,(v - u) \ .
\end{aligned}
$$

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Subgradient Algorithm

$$u_{ij}^{h+1} := u_{ij}^{h} + \alpha_h \left(\sum_{\ell=1}^{k} (x_{ij}^{\ell} + x_{ji}^{\ell}) - 1 \right)$$

Theorem 2 (Polyak 1967)

Let $L^*(u)$ be concave and bounded from above. If the sequence of step-lengths $(\alpha_h)_{h \in \mathbb{N}}$ satisfies $\alpha_h > 0$, $\lim_{h \to \infty} \alpha_h = 0$, and $\sum_h \alpha_h = \infty$, then the subgradient method converges to the maximum.

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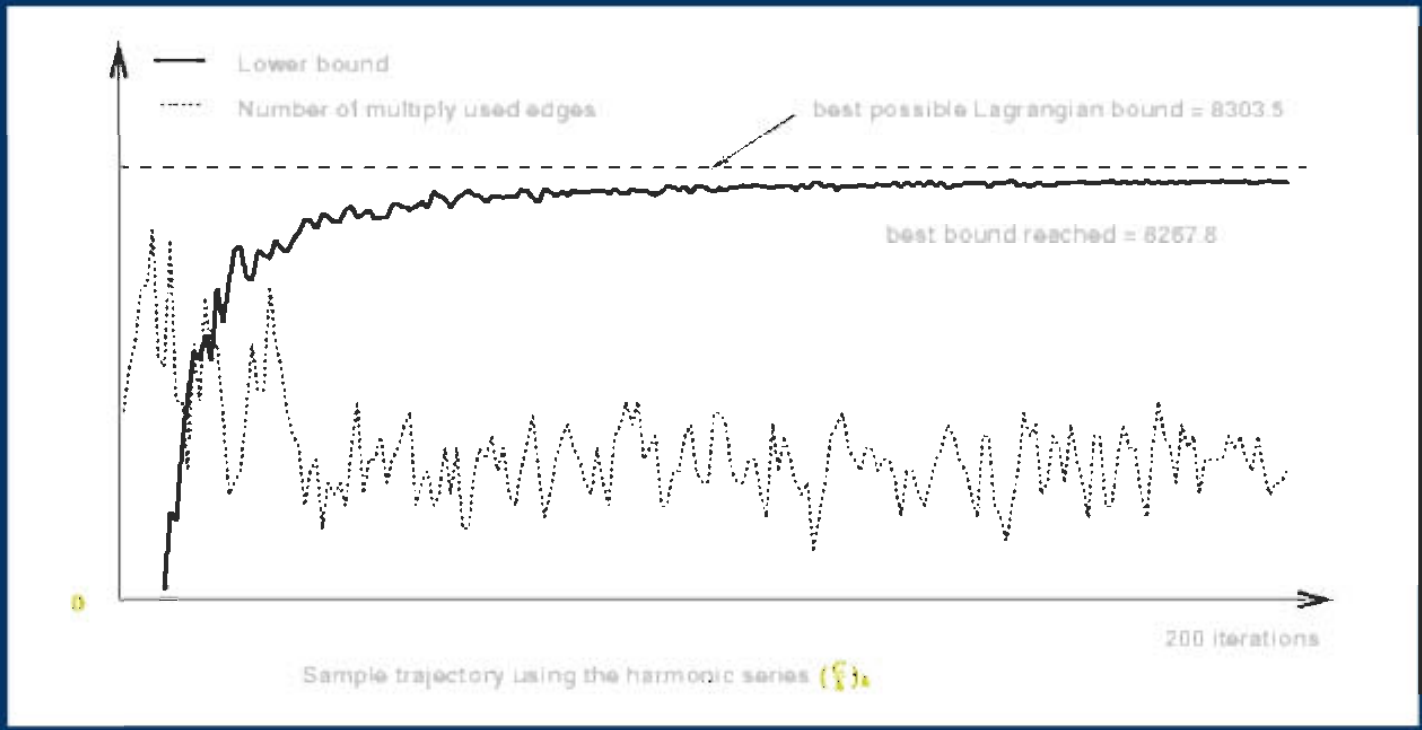
Lagrangian Relaxation

Typical Run



Sample trajectory using the harmonic series $\left(\frac{1}{k}\right)_k$.

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Lagrangian Relaxation

Practitioner's Choice

Use linear approximation $L(u) = cx(h) + u(Ax(h) - 1)$ to $L^*(u)$.

Let L^* be the optimum value of the Lagrangian dual.

Set θ_h such that
$L(u^{h+1}) = cx(h) + (u^h + \theta_h(Ax(h) - 1))(Ax(h) - 1) = L^*$.

Hence, $\theta_h = \frac{L^* - L^*(u^h)}{||Ax(h) - 1||^2}$.

Therefore,

$$\theta_h = \frac{\lambda_h(\text{UB} - L^*(u^h)}{||Ax(h) - 1||^2} \ .$$

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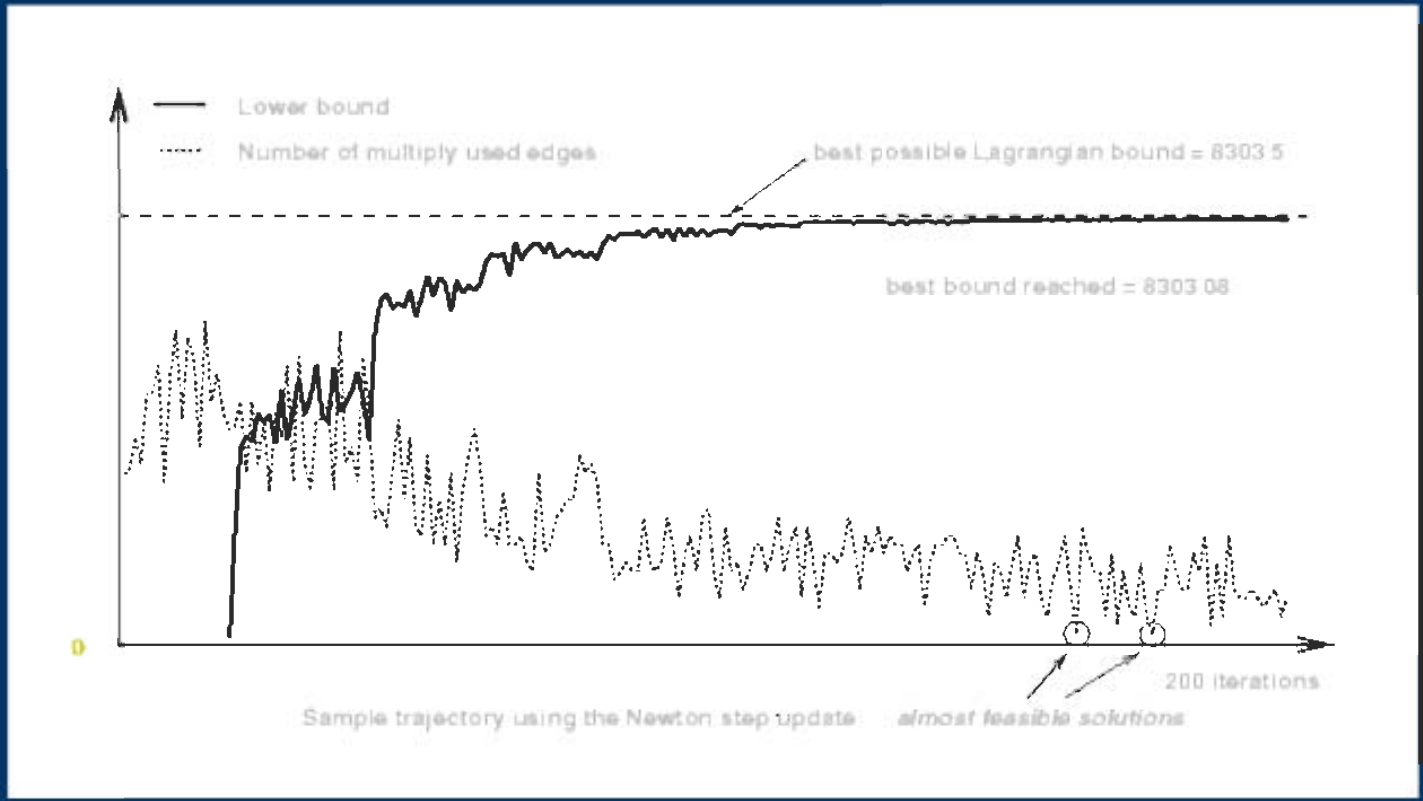
Lagrangian Relaxation

Typical Run



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Computational Results

Comparison of Lower Bounds

time[sec]	cheapest paths		Lagrangian		LP relaxation		optimum	
	bound	time	bound	time	bound	time	value	time
ger17e	54215	0.0	56336.5	1.3	56340.5	208	56467	390
ger17n	54679	0.0	58949.0	0.3	58949.0	321	58949	321
ger73e	88969	0.0	92132.0	1.7	92132.0	11198	92132	11198
ger73n	89601	0.0	98146.0	1.1	98146.0	502	98146	502
ger16e	53321	0.0	55265.8	2.9	55266.0	1771	55266	1771
ger16n	53423	0.0	57024.2	2.4	57086.0	1323	57086	1323
ger28e	80356	0.0	83094.8	4.6	83095.0	5634	83095	5634
ger28n	80125	0.0	87348.0	5.3	87404.4	8903	87498	9108
usa7e	60810	0.0	61498.8	0.6	61499.0	11	61499	11
usa7n	61214	0.0	66354.0	0.6	66398.0	8	69942	168
usa12e	67835	0.0	69898.9	0.5	69899.0	68	69899	162
usa12n	67835	0.0	73092.4	0.7	73104.7	43	74556	121
usa26e	104293	0.0	109493.7	3.9	109494	5029	109494	5029
usa26n	105108	0.0	123412.4	3.5	123601.1	2659	123835	5997
usa37e	105412	0.0	112486.0	2.1	112486.0	18036	112486	18252
usa37n	105412	0.0	121550.0	2.0	121550.0	10961	121550	10961

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Comparison of Lower Bounds

Theorem 3

$$\max_{u \geq 0} L^*(u) \geq LP \text{ value} .$$

Equality holds if the polyhedron defined by $Nx = b$, $0 \leq x \leq 1$ *is integer.*

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Comparison of Lower Bounds

Lagrange vs. LP

Proof...

$$\max_{u \geq 0} L^*(u) = \max_{u \geq 0} \min_{Nx=b, x \in \{0,1\}} L(x,u)$$

$$\geq \max_{u \geq 0} \min_{Nx=b, x \geq 0} L(x,u)$$

$$= \max_{u \geq 0} \min_{Nx=b, x \geq 0} (-u1 + (c + uA)x)$$

$$= \max_{u \geq 0} (-u1 + \min_{Nx=b, x \geq 0} (c + uA)x)$$

$$= \max_{u \geq 0} (-u1 + \max_{yN \leq c+uA} yb) = \cdots$$

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Comparison of Lower Bounds

Lagrange vs. LP

...Proof

$$\cdots = \max_{yN - uA \le c, u \ge 0} (-u1 + yb)$$

$$= \min_{Nx = b, Ax \le 1, x \ge 0} cx$$

$$=$$ optimal LP value.

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Branch&Bound

Ingredients

- Branching Strategy — [see picture]
- Exploration Strategy — Best Bound Search.
- Lower Bounds — Lagrangian Dual.
- Upper Bounds — Lagrangian Dual !!!

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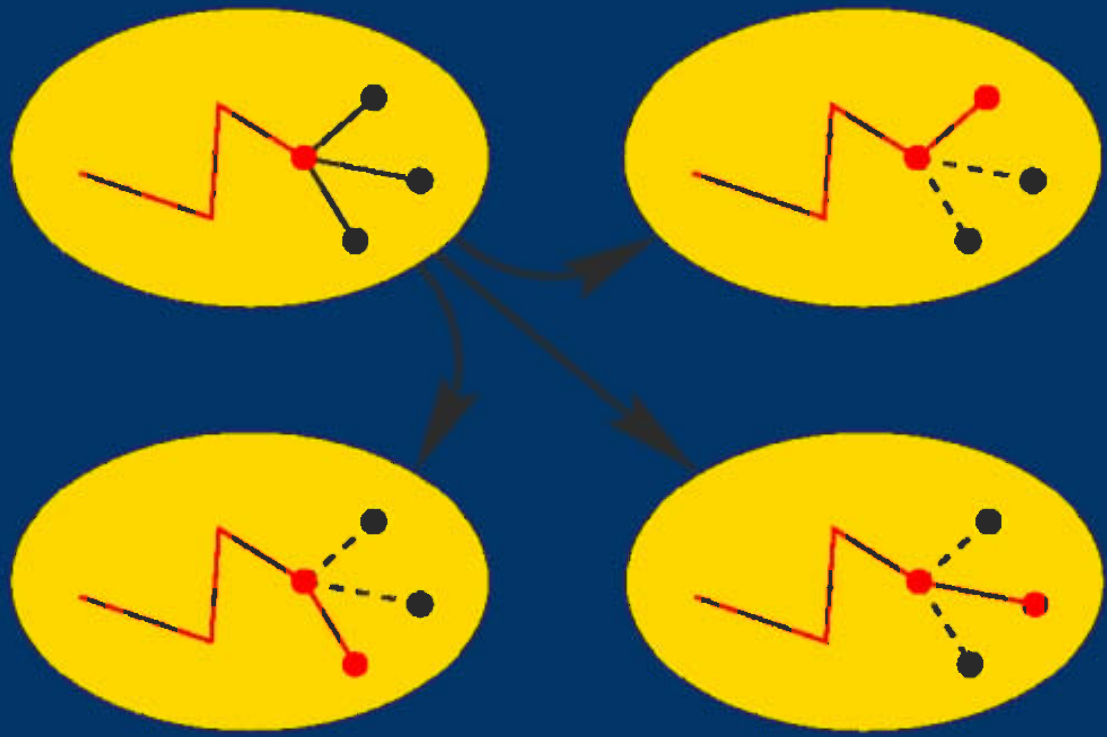
Branch&Bound

Problemspecific Branching



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Branch&Bound vs. CPLEX 3.0

Computational Results 1996

time[min]	size	branch&bound	CPLEX
ger17e	164	7.1	6.5
ger17n	nodes	0.6	5.4
ger73e	386	0.2	186.6
ger73n	edges	0.0	8.4
ger16e	434	8.2	29.5
ger16n	nodes	33.8	22.1
ger28e	978	13.5	93.9
ger28n	edges	94.8	151.8
usa7e	176	0.2	0.2
usa7n	nodes	0.6	2.8
usa12e	314	0.1	2.7
usa12n	edges	0.1	2.0
usa26e	617	0.1	83.8
usa26n	nodes	2.3	100.0
usa37e	1039	1.8	304.2
usa37n	edges	0.0	182.7

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Branch&Bound vs. CPLEX 6.5.3

Computational Results 2000

time[sec]	size	branch&bound I	branch&bound II	CPLEX
base1e	176 nodes	0.2	0.2	0.5
base1n	314 edges	196.1	17.65	9.6
base2e	185 nodes	0.7	0.3	1.0
base2n	354 edges	21.7	6.1	8.2
base3e	164 nodes	78.2	6.9	1.3
base3n	368 edges	5.2	3.9	7.3
base4e	434 nodes	314.3	20.5	5.2
base4n	978 edges	152.0	28.3	19.9
base5e	617 nodes	3.5	1.8	5.1
base5n	1039 edges	102.8	9.7	43.6
base6e	6894 nodes	15.0	8.1	
base6n	8072 edges		14.8	

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Reflection

Why was it successful?

- On the technical side:
  - Preprocessing
  - Pruning by infeasibility
  - Recycling of Lagrangian multipliers
  - Efficient solution of shortest paths problems
  - Data structures
- Project in itself:
  - Very good team in Berlin with complementary skills.
  - Competent partners at Deutsche Telekom.
  - Got data early enough.

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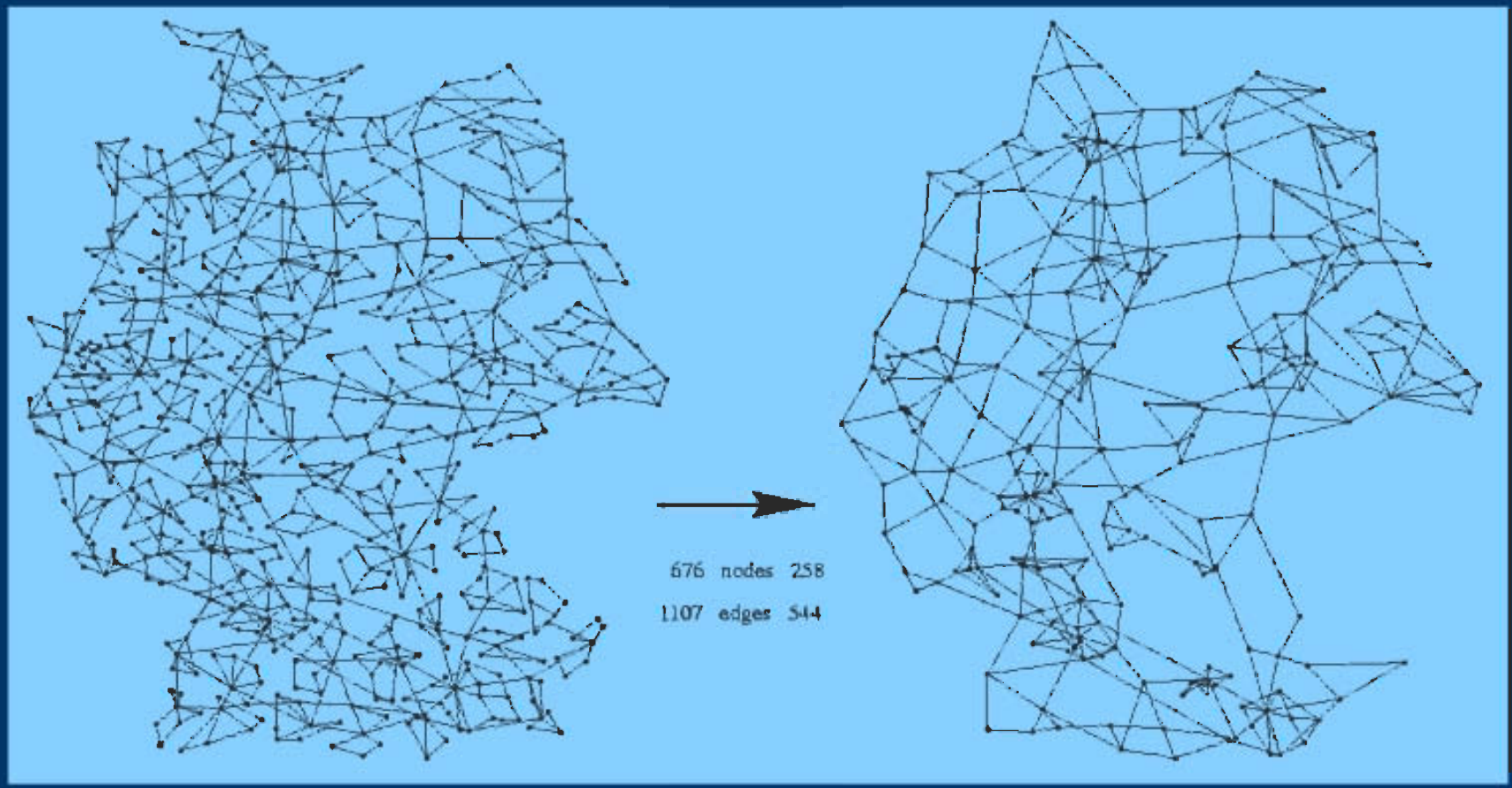
Preprocessing



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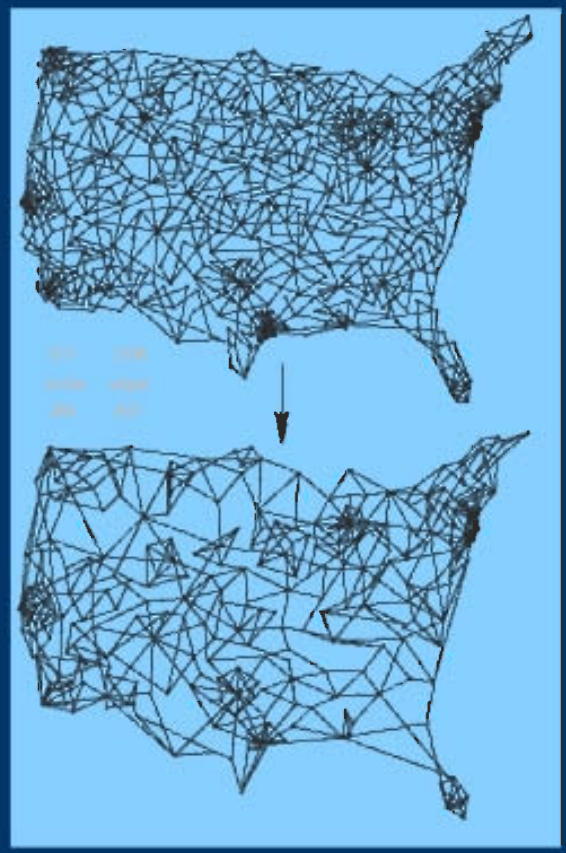
Preprocessing



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Done?

Did we solve the right problem?

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Extensions

- Edge capacities.
  - Dependent edges.
    - Varying bandwidth requirements.
      - Simultaneous embedding of many VPNs.

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Typical Steps

- Problem identification.
  - Problem penetration.
    - Modeling.
      - Problem complexity.
      - Solution approach.
    - Implementation.
  - Fine Tuning.
- Feedback.

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Some of Today's Lessons

- Good (= fast **and** close) lower bounds pay off.
- Tradeoff Lagrangian relaxation vs. LP relaxation.
- Algorithm fine tuning involves mathematical insights as well as engineering.
- Commercial software becomes increasingly stronger.

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