15.433 INVESTMENTS

Class 10: Equity Options

Part 1: Pricing

Spring 2003

SPX S&P 500 Index Options

Symbol: SPX Underlying: The Standard & Poor's 500 Index is a capitalization-weighted index of 500 stocks from a broad range of industries. The component stocks are weighted according to the total market value of their out-standing shares. The impact of a component's price change is proportional to the issue's total market value, which is the share price times the number of shares out-standing. These are summed for all 500 stocks and divided by a predetermined base value. The base value for the S&P 500 Index is adjusted to reflect changes in capitalization resulting from mergers, acquisitions, stock rights, substitutions, etc.

Multiplier: \$100.

Strike Price Intervals: Five points. 25-point intervals for far months.

Strike (Exercise) Prices: In-,at- and out-of-the-money strike prices are initially listed. New series are generally added when the underlying trades through the highest or lowest strike price available.

Premium Quotation: Stated in decimals. One point equals \$100. Minimum tick for options trading below 3.00 is 0.05 (\$5.00) and for all other series, 0.10 (\$10.00).

Expiration Date: Saturday immediately following the third Friday of the expiration month.

Expiration Months: Three near-term months followed by three additional months from the March quarterly cycle (March, June, September and December).

Exercise Style: European - SPX options generally may be exercised only on the last business day be-fore expiration.

Settlement of Option Exercise: The exercise-settlement value, SET, is calculated using the opening (first) reported sales price in the primary market of each component stock on the last business day (usually a Friday) before the expiration date. If a stock in the index does not open on the day on which the exercise & settlement value is determined, the last reported sales price in the primary market will be used in calculating the exercise-settlement value. The exercise-settlement amount is equal to the difference between the exercise-settlement value, SET, and the exercise price of the option, multiplied by \$100. Exercise will result in delivery of cash on the business day following expiration.

Position and Exercise Limits: No position and exercise limits are in effect. Each member (other than a market-maker) or member organization that maintains an end of day position in excess of 100,000 contracts in SPX (10 SPX LEAPS equals 1 SPX full value contract) for its proprietary account or for the account of a customer, shall report certain information to the Department of Market Regulation. The member must report information as to whether such position is hedged and, if so, a description of the hedge employed. A report must be filed when an account initially meets the aforementioned applicable threshold. Thereafter, a report must be filed for each incremental increase of 25,000 contracts. Reductions in an options position do not need to be reported. However, any significant change to the hedge must be reported.

Margin: Purchases of puts or calls with 9 months or less until expiration must be paid for in full. Writers of uncovered puts or calls must deposit / maintain 100*CUSIP Number:* 648815

Last Trading Day: Trading in SPX options will ordinarily cease on the business day (usually a Thursday) preceding the day on which the exercise-settlement value is calculated.

Trading Hours: 8:30 a.m.- 3:15 p.m. Central Time (Chicago time).

Equity Options: The Basics

A European style call option is the right to purchase, on a future date T, one unit of the underlying asset at a pre determined price K:

- At time 0 (today), pay C.
- At time T (the expiration date), exercise the option and get $(S_T - K)+$.

A put option gives the right to sell:

- At time 0 (today), pay P.
- At time T, get $(K - S_T)+$.

Most of the index options (e.g., SPX, QEX, NDX) are European style. All of the options on the individual stocks are American style, which allow for exercise before the expiration date T.

S&P 500 Options

.SPX (CBOE),1118.49,-1.82,
Nov 12 2001 @ 15:27 ET (Data 15 Minutes Delayed),

Calls	Last Sale	Net	Bid	Ask	Vol	Open Int	Puts	Last Sale	Net	Bid	Ask	Vol	Open Int
02 Mar 750.0 (SPZ CJ-E)	318.40	pc	370.10	374.10	-	32	02 Mar 750.0 (SPZ OJ-E)	4.80	1.10	3.10	4.50	1.00	13,629
02 Mar 775.0 (SPZ CO-E)	-	pc	346.20	350.20	-	-	02 Mar 775.0 (SPZ OO-E)	4.00	pc	4.00	5.30	-	12,283
02 Mar 800.0 (SPX CT-E)	-	pc	322.30	326.30	-	-	02 Mar 800.0 (SPX OT-E)	5.00	pc	5.00	6.30	-	6,600
02 Mar 850.0 (SPX CJ-E)	-	pc	275.50	279.50	-	-	02 Mar 850.0 (SPX OJ-E)	9.00	1.50	7.40	9.40	3.00	3,801
02 Mar 900.0 (SXB CT-E)	189.00	pc	229.80	233.80	-	51	02 Mar 900.0 (SXB OT-E)	12.50	1.10	10.90	13.80	7.00	13,075
02 Mar 925.0 (SXB CE-E)	-	pc	207.40	211.40	-	-	02 Mar 925.0 (SXB OE-E)	23.00	pc	13.20	16.20	-	633
02 Mar 950.0 (SXB CJ-E)	144.90	pc	185.70	189.70	-	1,078	02 Mar 950.0 (SXB OJ-E)	17.00	pc	16.30	19.30	-	7,719
02 Mar 975.0 (SXB CO-E)	153.00	61.00	164.40	168.40	2.00	2	02 Mar 975.0 (SXB OO-E)	25.00	pc	19.70	22.70	-	3,991
02 Mar 995.0 (SXB CS-E)	148.00	pc	148.00	152.00	-	2,109	02 Mar 995.0 (SXB OS-E)	24.00	0.60	22.80	26.80	2.00	11,110
02 Mar 1025. (SPQ CE-E)	94.00	pc	124.80	128.80	-	1,818	02 Mar 1025. (SPQ OE-E)	32.20	2.20	28.80	32.80	110.00	9,778
02 Mar 1050. (SPQ CJ-E)	108.00	2.50	106.20	110.20	4.00	8,059	02 Mar 1050. (SPQ OJ-E)	40.00	3.50	35.10	39.00	626.00	8,300
02 Mar 1075. (SPQ CO-E)	80.00	(14.00)	88.80	92.80	2.00	3,929	02 Mar 1075. (SPQ OO-E)	48.80	6.50	42.40	46.40	101.00	4,833
02 Mar 1100. (SPT CT-E)	72.00	(1.00)	72.30	76.30	560.00	13,293	02 Mar 1100. (SPT OT-E)	55.00	4.00	51.00	55.00	103.00	5,684
02 Mar 1125. (SPT CE-E)	53.00	(6.00)	57.50	61.50	1.00	7,351	02 Mar 1125. (SPT OE-E)	70.00	8.00	62.00	64.90	436.00	3,239
02 Mar 1150. (SPT CJ-E)	47.40	pc	44.80	48.80	-	8,354	02 Mar 1150. (SPT OJ-E)	79.00	5.00	73.00	77.00	1.00	9,298
02 Mar 1175. (SPT CO-E)	29.50	(7.50)	33.70	37.70	3.00	3,148	02 Mar 1175. (SPT OO-E)	93.00	pc	86.70	90.70	-	4,549
02 Mar 1200. (SZP CT-E)	21.00	(5.30)	24.50	28.50	3.00	13,211	02 Mar 1200. (SZP OT-E)	97.60	pc	102.80	106.80	-	8,920
02 Mar 1225. (SZP CE-E)	19.00	pc	17.60	20.60	-	3,922	02 Mar 1225. (SZP OE-E)	140.00	18.00	120.20	124.20	5.00	3,514
02 Mar 1250. (SZP CJ-E)	16.50	pc	11.80	14.80	-	10,727	02 Mar 1250. (SZP OJ-E)	143.00	(15.00)	139.10	143.10	1.00	4,579
02 Mar 1275. (SZP CO-E)	8.50	2.50	8.00	10.00	1.00	3,035	02 Mar 1275. (SZP OO-E)	193.00	pc	159.60	163.60	-	1,080
02 Mar 1280. (SZP CP-E)	6.30	pc	7.30	9.30	-	3	02 Mar 1280. (SZP OP-E)	72.50	pc	163.70	167.70	-	2
02 Mar 1300. (SXY CT-E)	5.40	pc	5.00	6.40	-	6,073	02 Mar 1300. (SXY OT-E)	195.00	16.80	181.10	185.10	200.00	2,007
02 Mar 1325. (SXY CE-E)	5.00	pc	3.10	4.50	-	2,241	02 Mar 1325. (SXY OE-E)	328.00	pc	203.80	207.80	-	213
02 Mar 1350. (SXY CJ-E)	3.50	pc	1.90	2.80	-	4,957	02 Mar 1350. (SXY OJ-E)	250.50	pc	227.20	231.20	-	918
02 Mar 1375. (SXY CO-E)	1.30	pc	1.00	1.90	-	789	02 Mar 1375. (SXY OO-E)	276.00	pc	251.10	255.10	-	266
02 Mar 1400. (SXZ CT-E)	1.00	pc	0.45	1.30	-	8,718	02 Mar 1400. (SXZ OT-E)	290.00	19.00	275.30	279.30	200.00	1,230
02 Mar 1425. (SXZ CE-E)	0.80	pc	0.05	0.95	-	269	02 Mar 1425. (SXZ OE-E)	-	pc	299.70	303.70	-	-
02 Mar 1450. (SXZ CJ-E)	0.35	pc	-	0.90	-	1,986	02 Mar 1450. (SXZ OJ-E)	344.00	pc	324.30	328.30	-	679
02 Mar 1475. (SXZ CO-E)	2.00	pc	-	0.90	-	94	02 Mar 1475. (SXZ OO-E)	343.00	pc	349.10	353.10	-	60
02 Mar 1500. (SXM CT-E)	0.50	pc	-	0.90	-	1,173	02 Mar 1500. (SXM OT-E)	408.50	pc	373.90	377.90	-	55
02 Mar 1550. (SXM CJ-E)	0.50	pc	-	0.90	-	28	02 Mar 1550. (SXM OJ-E)	-	pc	423.30	427.30	-	-
02 Mar 1600. (SPB CT-E)	0.55	pc	-	0.90	-	931	02 Mar 1600. (SPB OT-E)	-	pc	472.90	476.90	-	-
02 Apr 1020. (SYU DD-E)	-	pc	-	-	-	-	02 Apr 1020. (SYU PD-E)	-	pc	-	-	-	-
02 Jun 700.0 (SPZ FT-E)	-	pc	420.80	424.80	-	-	02 Jun 700.0 (SPZ RT-E)	7.40	pc	4.50	5.90	-	734
02 Jun 725.0 (SPZ FE-E)	-	pc	397.40	401.40	-	-	02 Jun 725.0 (SPZ RE-E)	11.00	pc	5.40	7.40	-	14
02 Jun 750.0 (SPZ FJ-E)	-	pc	374.00	378.00	-	-	02 Jun 750.0 (SPZ RJ-E)	7.30	pc	6.70	8.70	-	1,064
02 Jun 800.0 (SPX FT-E)	-	pc	327.70	331.70	-	-	02 Jun 800.0 (SPX RT-E)	10.00	pc	9.70	11.70	-	2,852
02 Jun 850.0 (SPX FJ-E)	225.00	pc	282.80	286.80	-	121	02 Jun 850.0 (SPX RJ-E)	14.50	pc	13.50	16.50	-	2,993
02 Jun 900.0 (SXB FT-E)	-	pc	239.30	243.30	-	-	02 Jun 900.0 (SXB RT-E)	19.50	pc	19.30	22.30	-	6,983
02 Jun 950.0 (SXB FJ-E)	161.50	pc	197.80	201.80	-	28	02 Jun 950.0 (SXB RJ-E)	29.00	(1.00)	26.60	30.60	3.00	4,332
02 Jun 995.0 (SXB FS-E)	164.00	pc	162.60	166.60	-	2,670	02 Jun 995.0 (SXB RS-E)	37.50	pc	35.70	39.70	-	9,137
02 Jun 1025. (SPQ FE-E)	121.00	pc	140.50	144.50	-	1,358	02 Jun 1025. (SPQ RE-E)	44.00	pc	43.20	47.20	-	6,127
02 Jun 1045. (SPQ FI-E)	-	pc	-	-	-	1,350	02 Jun 1045. (SPQ RI-E)	-	pc	-	-	-	1,350
02 Jun 1050. (SPQ FJ-E)	115.00	(10.30)	123.10	127.10	5.00	3,393	02 Jun 1050. (SPQ RJ-E)	52.00	pc	50.40	54.40	-	12,129
02 Jun 1075. (SPQ FO-E)	108.50	pc	106.70	110.70	-	1,186	02 Jun 1075. (SPQ RO-E)	66.40	6.40	58.60	62.60	2.00	1,144
02 Jun 1100. (SPT FT-E)	91.30	(3.00)	91.60	95.60	401.00	5,411	02 Jun 1100. (SPT RT-E)	76.00	6.00	68.20	72.20	10.00	14,313
02 Jun 1125. (SPT FE-E)	79.00	pc	77.60	81.60	-	1,885	02 Jun 1125. (SPT RE-E)	84.50	3.50	78.80	82.80	3.00	1,412
02 Jun 1150. (SPT FJ-E)	61.00	(6.00)	64.80	68.80	5.00	5,415	02 Jun 1150. (SPT RJ-E)	103.00	pc	90.60	94.60	-	5,536
02 Jun 1175. (SPT FO-E)	54.00	pc	53.10	57.10	-	673	02 Jun 1175. (SPT RO-E)	86.00	pc	103.60	107.60	-	160
02 Jun 1200. (SZP FT-E)	46.50	pc	42.40	46.40	-	7,255	02 Jun 1200. (SZP RT-E)	150.00	pc	117.50	121.50	-	6,462
02 Jun 1225. (SZP FE-E)	35.50	pc	33.90	37.90	-	1,451	02 Jun 1225. (SZP RE-E)	179.50	pc	133.60	137.60	-	446
02 Jun 1250. (SZP FJ-E)	27.50	pc	26.20	30.20	-	6,959	02 Jun 1250. (SZP RJ-E)	153.00	pc	150.60	154.60	-	2,280
02 Jun 1300. (SXY FT-E)	17.50	pc	15.00	18.00	-	7,065	02 Jun 1300. (SXY RT-E)	244.90	pc	188.20	192.20	-	4,474
02 Jun 1325. (SXY FE-E)	11.20	pc	10.80	13.80	-	1,469	02 Jun 1325. (SXY RE-E)	227.00	pc	208.70	212.70	-	152
02 Jun 1350. (SXY FJ-E)	7.00	(1.00)	8.10	10.10	11.00	5,481	02 Jun 1350. (SXY RJ-E)	248.00	pc	230.10	234.10	-	2,724
02 Jun 1375. (SXY FO-E)	6.00	pc	5.60	7.60	-	787	02 Jun 1375. (SXY RO-E)	280.00	pc	252.20	256.20	-	5
02 Jun 1400. (SXZ FT-E)	4.00	--	4.00	5.40	10.00	12,350	02 Jun 1400. (SXZ RT-E)	290.00	1.00	275.00	279.00	200.00	5,677
02 Jun 1425. (SXZ FE-E)	1.80	pc	2.90	3.90	-	329	02 Jun 1425. (SXZ RE-E)	-	pc	298.20	302.20	-	-
02 Jun 1450. (SXZ FJ-E)	2.20	pc	1.90	2.80	-	8,703	02 Jun 1450. (SXZ RJ-E)	388.00	pc	321.90	325.90	-	3,946
02 Jun 1475. (SXZ FO-E)	1.90	pc	1.25	2.15	-	123	02 Jun 1475. (SXZ RO-E)	-	pc	345.90	349.90	-	-
02 Jun 1500. (SXM FT-E)	1.20	pc	0.60	1.50	-	8,537	02 Jun 1500. (SXM RT-E)	396.00	pc	369.80	373.80	-	2,882
02 Jun 1525. (SXM FE-E)	0.90	pc	0.25	1.15	-	320	02 Jun 1525. (SXM RE-E)	347.00	pc	394.10	398.10	-	101
02 Jun 1550. (SXM FG-E)	-	pc	-	-	-	-	02 Jun 1550. (SXM RG-E)	-	pc	-	-	-	-
02 Jun 1550. (SXM FJ-E)	0.30	--	0.10	0.95	5.00	3,458	02 Jun 1550. (SXM RJ-E)	417.00	pc	418.60	422.60	-	1,622
02 Jun 1600. (SPB FT-E)	0.40	pc	-	0.90	-	6,212	02 Jun 1600. (SPB RT-E)	302.00	pc	467.60	471.60	-	944
02 Jun 1650. (SPB FJ-E)	2.90	pc	-	0.90	-	3,849	02 Jun 1650. (SPB RJ-E)	513.00	pc	516.80	520.80	-	350
02 Jun 1700. (SPV FT-E)	0.10	pc	-	0.90	-	3,439	02 Jun 1700. (SPV RT-E)	590.00	pc	566.00	570.00	-	248
02 Jun 1750. (SPV FJ-E)	0.65	pc	-	0.90	-	1,120	02 Jun 1750. (SPV RJ-E)	724.70	pc	615.20	619.20	-	43
02 Jun 1800. (SYV FT-E)	0.05	pc	-	0.90	-	4,132	02 Jun 1800. (SYV RT-E)	811.00	pc	664.50	668.50	-	596

Figure 2: Source: www.cboe.com

The Value of a Call Option

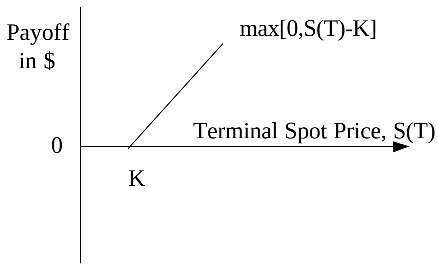

Figure 3: Value of a Long Call at Expiration

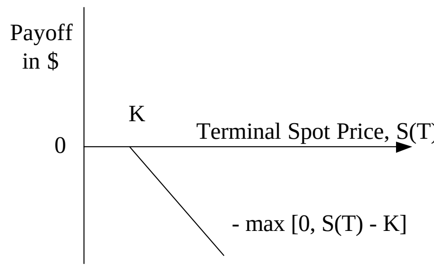

Figure 4: Value of Short Call at Expiration

Notice that the short call option payoff is unbounded from below.

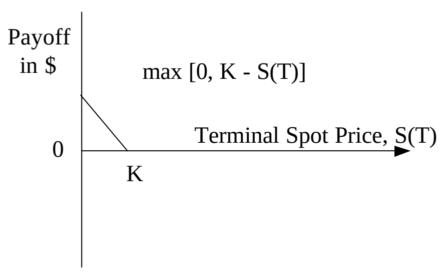

Figure 5: Value of a Long Put at Expiration

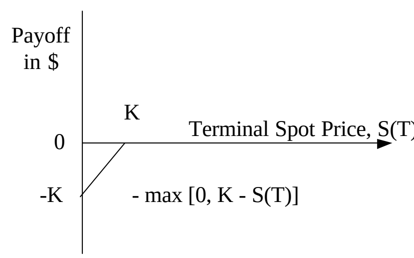

Figure 6: Value of a Short Put at Expiration

Hence you can buy a put option if you are a pessimist, i.e., you have a hunch, but are not absolutely certain, about the stock price, S, going below the strike price, K, at maturity (for Euro-pean options).

Notice that the long put option payoff is bounded above by K.

Option Pricing

"Suppose the underlying security does not pay dividend,

$$r_T = ln\left(S_T\right) - ln\left(S_O\right) \tag{1}$$

or equivalently:

$$S_0 = E\left(e^{-r_{i,T}} \cdot S_T\right) \tag{2}$$

A call option paying $(S_T - K)^+$ at time T must be worth:

$$C_0 = E\left(e^{-r_{i,T}} \cdot (S_T - K)^+\right) \tag{3}$$

From the intuition of the CAPM, we know that the above evaluation depends on the risk aversion coefficient A of the representative investor. "For example, if the underlying asset is the market portfolio, then

$$E\left(r_T\right) - r_f = \bar{A} \cdot var\left(r_T\right) \tag{4}$$

The question is: how to evaluate

$$C_0 = E\left(e^{-r_{i,T}} \cdot (S_T - K)^+\right) \tag{5}$$

Single Period Binominal-Valuation Problem

Suppose that a European call option struck at \$ 50 matures in one year ($t = 1$). The riskless rate per year is 25%, compounded annually, so that one dollar invested at the riskless rate grows to \$ 1.25 over the year. The stock underlying the option pays no dividends over the year and its current price of \$ 40 will either double or halve over the year. Assuming frictionless markets and no arbitrage, what must be the current price of the call, C_0?

- Strike Price of the call = K = \$ 50
- Riskless return over the period = R(1) = 1.25
- Unit Bond price = $B(0,1) = \frac{1}{1.25} = \$0.80$
- Stock price at the beginning of the period = $S_0 = \$40$
- Up return of the stock = $U = 2$
- Down return of the stock = $D = \frac{1}{2}$
- Call value at the end of the period given an Up return of the stock = $C_U = max[0, S_0 \cdot U - K] = \$30$
- Call value at the end of the period given a down return of the stock = $C_D = max[0, S_0 \cdot D - K] = \$0$
- Call value at the beginning of the period = $C_0 =?$ (Ans: \$ 12. The next section will derive the price).

Spanning the Payoffs

Consider a portfolio of m_0 shares of the underlying stock and B_0 dollars invested in the riskless asset, Assume you form the portfolio at time 0. The portfolio has two possible values at the end of the period, depending on whether the stock price goes up or down:

$$Up : m_0 \cdot 80 + B_0 \cdot 1.25 \tag{6}$$

$$Down : m_0 \cdot 20 + B_0 \cdot 1.25 \tag{7}$$

Similarly, the call value has two possible values:

$$\text{Up: 30 or Down: 0}$$

We can choose the number of shares, m_0, and the amount invested in the riskless asset, B_0, today, so that the value of the stock-bond portfolio equates to the value of the call next year:

1. $m_0 \cdot 80 + B_0 \cdot 1.25 = 30$
2. $m_0 \cdot 20 + B_0 \cdot 1.25 = 0$

Subtracting (2) from (1) and solving for the number of shares, m_0, we get

$$m_0 = \frac{30 - 0}{80 - 20} = \frac{1}{2} \tag{8}$$

Plugging for m_0 into either (1) or (2), we get $B_0 = -8\$$. The minus sign for B_0 means we have to borrow \$8 at the riskless rate.

Valuation

Since buying $\frac{1}{2}$ of a share of the stock and shorting \$8 of the risk-less asset duplicates the payoff of the all, avoiding arbitrage requires that the current price of the traded call equals the cost of duplicating it. Hence, the current call value is: $V(0) = \frac{1}{2} \cdot \$40 - \$8 = \$12$.

Remarks:

- The required number of shares, $m_0 = \frac{1}{2}$, is the difference in next year's possible call values, expressed as a proportion of the difference in next year's possible stock prices:
$$m_0 = \frac{30 - 0}{80 - 20} = \frac{1}{2} \tag{9}$$

- For a European call option, the required number of shares is always between 0 and 1. In a graph of call values against stock prices, the required number of shares is the slope of the graph. For this reason, the required number of shares is often called the delta of the call. Delta is the sensitivity of the call option price to stock price changes. It is a very useful parameter in hedging. The amount invested in the riskless asset is negative ($B_0 = -\$8$). Consequently, the riskless bond must be shorted. Short selling bonds is equivalent to borrowing. The amount to be repaid at the end of the period is $(B_0 \cdot 1.25) = \$10$, irrespective of being in the up state or down state.

- This approach can be used in a binomial framework to value any claim whose payoff is contingent on the price of the stock (e.g., a put).

- Notice that the probabilities q and $1-q$ do not enter the valuation argument at all. Hence, 2 individuals who disagree about the probabilities of outcomes $(q, 1-q)$, but who agree on the outcomes (i.e., U and D), will agree that the traded option's value today is 12.

- Since we assume that the stock can only take on two values from any node, we only need two assets (the underlyer and the riskless asset) to replicate the payoff of the option. Any additional uncertainty (example, stochastic interest rates) would require additional assets.

Fisher Black on Option Pricing

"I applied the Capital Asset Pricing Model to every moment in a warrant's life, for every possible stock price and warrant value ... I stared at the differential equation for many, many months. I made hundreds of silly mistakes that led me down blind alleys. Nothing worked . . .

[The calculations revealed that] the warrant value did not depend on the stock's expected return, or on any other asset's expected return. That fascinated me.

[He adds:] Then Myron-Scholes and I started working together".

Risk Neutral Pricing

It turns out that Black was not far from the truth!

- Two hypothetical investors: one risk neutral (A = 0), and one risk averse ($A > 0$).
- Suppose both investors are willing to pay the market price S_0 for the underlying asset.
  But their required rates of return are different:
  - For the risk neutral investor, it is simply the riskfree rate r_f.
  - For the risk averse, it is the riskfree rate r_f plus a positive risk premium.
- If they agreed on So, they would also agree on C_0 for the call. If this is true, the easiest way to get C_0 is to let the risk neutral investor do the pricing. Hence the term "risk neutral pricing."

One key assumption: All investors, regardless of their risk attitudes, agree on today's stock price S_0.

Risk-Neutral Valuation

Risk neutral valuation is a trick for valuing options quickly when investors can be risk-averse. It is not a model which assumes investors are all risk-neutral.
Recall that the version of the binomial model we studied assumed frictionless markets, no arbitrage, European options, a constant riskless rate, no payouts, and a multiplicative binomial process for the underlyer's spot price.
We were able to value a European call and put without any knowledge of investor preferences or beliefs regarding the likelihood of the up or down states. This is called *Valuation by Duplication*.
Since the same value results regardless of investor preferences, we can pretend that investors are risk-neutral as an aid in calculating values.
In this case, the expected return on the underlyer is the riskless return r_f. Letting π denote the risk-neutral probability of an up jump, we have:

$$S_t = \frac{1}{r_f} \cdot [\pi \cdot S_t \cdot U + (1-\pi) \cdot S_t \cdot D] \tag{10}$$

implying

$$\pi \cdot U + (1-\pi) \cdot D = r_f \tag{11}$$

or equivalently

$$\pi = \frac{r_f - D}{U - D} \tag{12}$$

Under risk-neutrality, the expected payoff of a 1-period call is:

$$c_1^+ = (1-\pi) \cdot c_1^- \tag{13}$$

where recall:
$c_1^+ = max\left[0, S_1^+ - K\right]$ is the call value in the up state
$c_1^- = max\left[0, S_1^- - K\right]$ is the call value in the down state
$\pi = \frac{r_f - D}{U - D}$ is the risk-neutral probability of an up jump found by equating the expected return from the underlying to the riskless return.

In a risk-neutral world, the expected return on an option is also the riskless return. Thus, the initial call value c_0 is given by discounting the above expected payoff at the riskless return r_f:

$$c_0 = r_f^{-1} \cdot \left[\pi \cdot c_1^+ + (1-\pi) \cdot c_1^-\right] \tag{14}$$

Valuation by Duplication is the reason why risk-neutral valuation works. The same idea can be used for puts and for multiple periods. Risk-neutral valuation permits very quick calculation of option values. Risk-neutral valuation, hence, simplifies the valuation problem (compared to valuation by duplication which was mighty cumbersome for multiple periods). Furthermore, the composition of the duplicating portfolio is easily calculated once the option values are known. Armed with the risk-neutral probabilities π and $1-\pi$, we can duplicate and price anything. For example, to duplicate a contingent claim paying X_U dollars if the spot price goes up and X_D dollars if the spot price goes down, we use risk-neutral valuation to calculate the current price of the contingent claim as:

$$r_f^{-1} \cdot [\pi \cdot X_U + (1-\pi) \cdot X_D] \tag{15}$$

which is interpreted as the discounted expected payoff under risk-neutrality. Example: Recall (see previous page) for 1-period calls,

$$c_0 = r_f^{-1} \cdot \left[\pi \cdot c_1^+ + (1-\pi) \cdot c_1^-\right] \tag{16}$$

where $X_U = C_1^+$ and $X_D = C_1^-$. If we think of X_U and X_D as option values in successor nodes, then we have a recipe for valuing in multiple periods as illustrated next. Note: The risk neutral probabilities π and $1-\pi$ are usually referred to also as the equivalent martingale probabilities.

Two Period Example

Given T $=2$, S$_0$ $=40$, $U=2$, $D=\frac{1}{2}$,r$_f$ $=\frac{5}{4}$, $K=50$, what is the risk-neutral probability of an up jump?

$$\pi = \frac{r_f - D}{U-D} = \frac{\frac{5}{4}-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1}{2} \tag{17}$$

Use risk-neutral valuation to calculate the values of a call at each node bow. Also give the number of stocks and the amount lent in order to replicate the call value at each node. Compare your answers with those obtained using the method of replication (see 2-period example in Section 3 of Overheads 3).

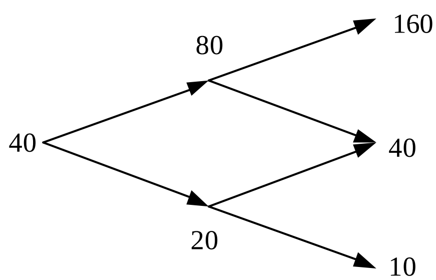

Figure 7 : Two period binomial tree, call option

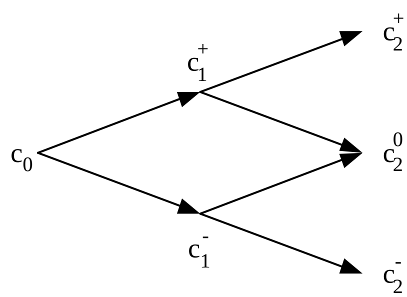

Figure 8: Two period binomial tree, call option

Answer: Note that $c_2^+ = max[0, 160-50] = 110$, while $c_2^0 = c_2^- = 0\$$.

$$\begin{aligned} c_1^+ &= \frac{1}{r_f}\cdot\left[\frac{1}{2}\cdot 110 + \frac{1}{2}\cdot 0\right] && (18)\\ c_1^- &= 0 && (19)\\ c_0 &= \frac{1}{r_f}\cdot\left[\frac{1}{2}\cdot c_1^+ + \frac{1}{2}\cdot c_1^-\right] && (20)\\ & && (21)\end{aligned}$$

The call's Delta and the amount lent at each node is given by:

$$\begin{aligned}
m_1^+ &= \frac{c_2^+ - c_2^0}{S_2^+ - S_2^0} = \frac{11}{12} & (22)\\
B_1^+ &= \frac{c_2^+ S_2^0 - c_2^0 S_2^+}{r_f\left[S_2^+ - S_2^0\right]} = -29\frac{1}{3} & (23)\\
m_0 &= \frac{c_1^+ - c_1^0}{S_1^+ - S_1^0} = \frac{11}{15} & (24)\\
B_0 &= \frac{c_1^+ S_1^0 - c_1^0 S_1^+}{r_f\left[S_1^+ - S_1^0\right]} = -11\frac{11}{15} & (25)
\end{aligned}$$

Use risk-neutral valuation to calculate the values of a put at each node bow. Also give the number of stocks and the amount lent in order to replicate the put value at each node.

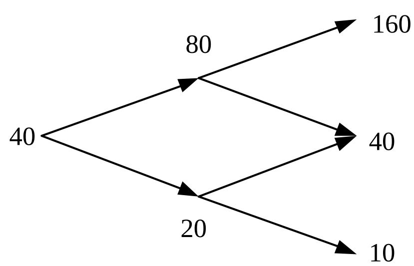

Figure 9: Two period binomial tree, put option

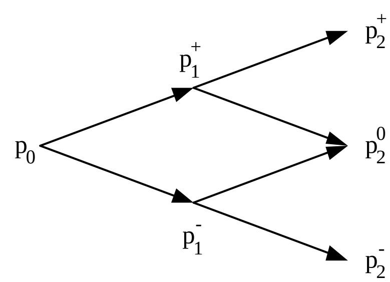

Figure 10: Two period binomial tree, put option

Answer: Note that $p_2^- = max[0, 50 - 10] = \$40$, while $p_2^0 = max[0, 50 - 40] = \$10$ and $p_2^+ = 0$. Therefore,

$$\begin{aligned}
p_1^+ &= \frac{1}{r_f} \cdot \left[\frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0\right] = \$4 & (26)\\
p_1^- &= \frac{1}{r_f} \cdot \left[\frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 40\right] = \$20 & (27)\\
p_0 &= \frac{1}{r_f} \cdot \left[\frac{1}{2} \cdot p_1^+ + \frac{1}{2} \cdot p_1^-\right] = \$9.60 & (28)
\end{aligned}$$

We can use Put-Call Parity to check whether our answers make sense. Recall $\mathrm{p}_0 + S_0 = c_0 + K \cdot B(0, 2)$.

$LHS = p_0 + S_0 = 9.60 + \ 40 = \$49.60$.

$RHS = c_0 + K \cdot B(0, 2) = \$17.60 + \$32 = \$49.60$.

What is the Intuition?

If there is only one source of uncertainty in the underlying stock, then the effect of the random shock is fully reflected in the underlying stock price.

In such a setting, option becomes redundant.

Investors are risk averse: they are worried about the systematic random fluctuations in the stock price. This fear is fully ex-pressed when investors price the underlying security.

When it comes to price options, investors are already at a comfortable level regarding risk and reward. Options, being redundant in this setting, provide no additional information about the risk or the reward.

There is no assumption about investors being risk neutral. Risk neutral pricing is a trick to simplify option pricing.

The Assumptions

1. Constant riskfree borrowing and lend rate r_f.

2. The underlying asset can be continuously traded with no transactions costs, no short sale constraints, perfectly divisible, and no taxes.

3. We assume no dividends, but this restriction can be relaxed.

4. The price of the underlying asset follows a Geometric Brownian Motion:

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma \cdot dB_t \tag{29}$$

This model is the continuous time version of the random walk model we have studied in Class 9. The increment ΔB_t of a Brownian motion is normally distributed with mean zero and variance Δt.

The Black Scholes Formula

A European call struck at K, expiring on date T.

$$C_0 = S_0 \cdot N\left(d_1\right) - e^{-rT} K N\left(d_2\right) \tag{30}$$

where

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r_f + \frac{\sigma^2}{2}\right) \cdot T}{\sigma \cdot \sqrt{T}} \tag{31}$$

$$d_2 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r_f - \frac{\sigma^2}{2}\right) \cdot T}{\sigma \cdot \sqrt{T}} \tag{32}$$

and where

- S_0 is the initial stock price,
- σ is its volatility
- r_f is the riskfree rate
  $N\left(d_x\right)$ is the probability that the outcome of a standard normal distribution is less than d.

Put/Call Parity

$$c - p = S_t^z - K \cdot B\,(t, T) \tag{33}$$

$$= max\,[0, S_t^z - K \cdot B\,(t, T)] - max\,[0, K \cdot B(t, T) - S_t^z]\,. \tag{34}$$

$$\boxed{c - p = S - e^{-r(T-T)} \cdot K \quad put - call\ parity}$$

The value of the call minus the value for the put is equal to the value for the stock minus the present value of K.

We have two investments:

1. Buy the call option and sell the put option. Value: $c - p$
2. Go long the stock and sell a riskless, zero-coupon bond maturing at time T to K. Value: $S - e^{-r(T-t)} \cdot K$

Neither of these instruments incur any costs during their lifetime. Let‘s examine their values at time T, starting with investment one, the long-call, short-put investment. Since the call and the put have the same strike, at expiration either the call will be in the money or the put will be in the money - but never both. Write S_T for the value for the stock at time T. If the call is in the money, the payoff is $S_T - K$, since the position is long. On the other hand, if the short put is in the money, the its payoff is $-(K - S_T) = S_T - K$. That is, since the position is short, its payoff is the negative of the usual $K - S_T$, independent of the stock price at Time T.

The second investment, the stock-bond portfolio is comparatively easy to value. At time T, the bond will have matured to a value of K, and therefore the long-stock, short-bond position will have a value of $S_T - K$. Both investments have the same value at time T, and moreover, cost nothing to maintain. Therefore, our basic arbitrage argument tells us the investments must have the same initial value, that is:

$$\underbrace{c - p}_{value\ of\ investment\ 1} = \underbrace{S - e^{-r(T-t)} \cdot K}_{value\ of\ investment\ 2} \tag{35}$$

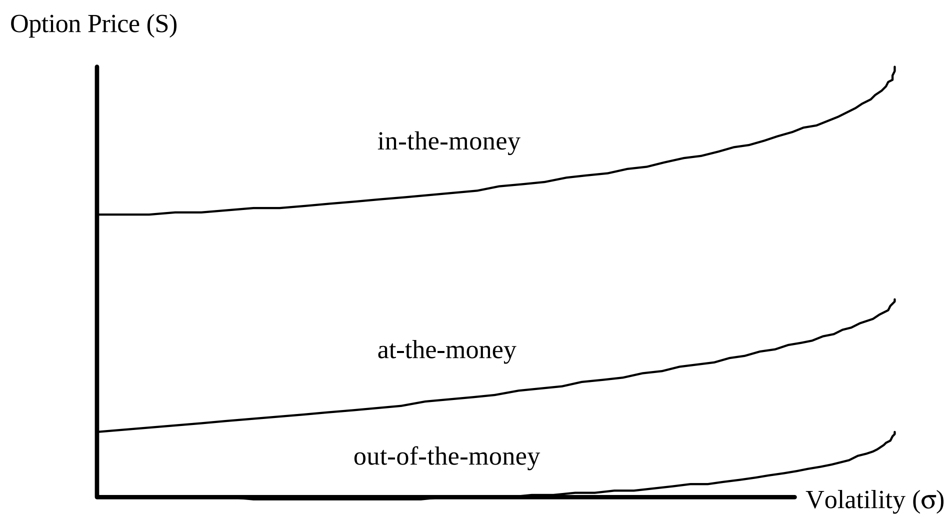

Figure 11: Volatility-levels for in, at or out-of-the-money options.

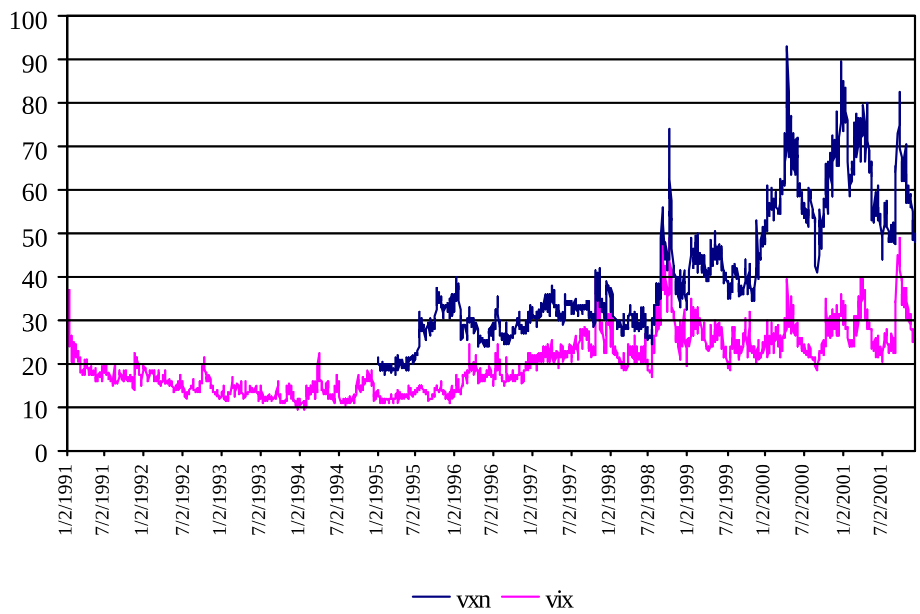

Figure 12: Distribution of the Nasdaq-index returns for the year 2000, data-source: Bloomberg.

VXN is based on the implied volatility of the Nasdaq-100 (NDX) options, while the VIX volatility is based on the implied volatilities of the S&P 100-OEX options.

The Options Market and Volatility

A casual inspection of the options market leads one to believe that stock volatility does not stay constant over time. Why?

Investors express their view on the future market volatility by trading options.

Effectively, the equity options market serves as an information central, collecting updates about the future market volatility:

- short dated options: near term volatility;
- long dated options: long term volatility.

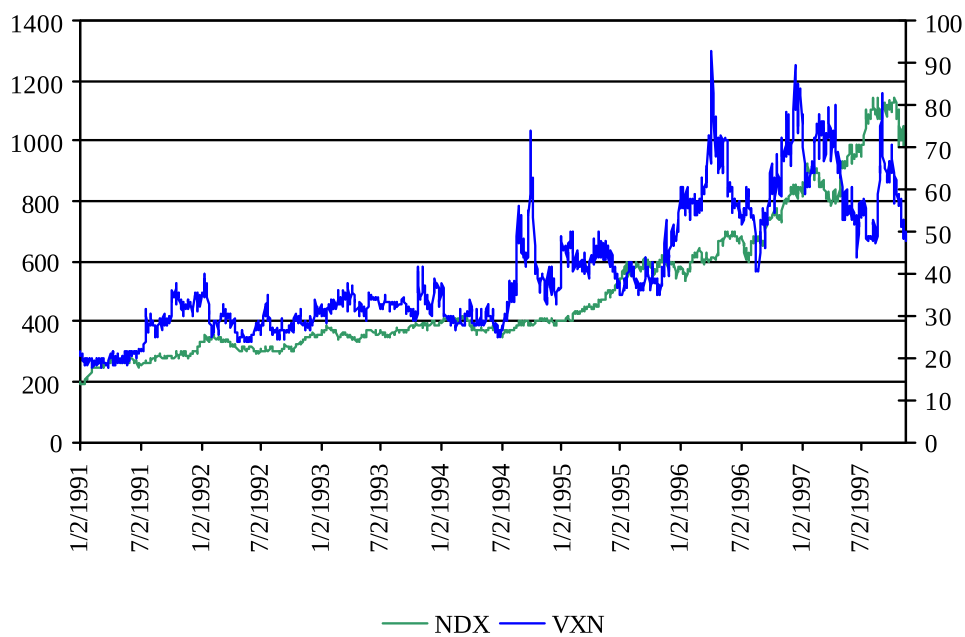

Figure 13: VXN: Impl. volatility of options on the NDX; NDX: Impl. volatility of Nasdaq-100 index, datasource: Bloomberg Professional.

Important: VXN and NDX can have somewhat negative correlations!

The Option Implied Volatility

At time 0, a call option struck at K and expiring on date T is traded at C_0. At the same time, the underlying stock price is traded at S_0, and the riskfree rate is r_f.

If we know the market volatility at time 0, we can apply the Black Scholes formula:

$$C_{BS}^0 = BS\left(S_0, K, T, \sigma, r_f\right) \tag{36}$$

Volatility is something that we don't observe directly. But using the market observed price C_0, we can back it out:

$$C_0 = BS\left(S_0, K, T, \sigma^I, r_f\right) \tag{37}$$

If the Black Scholes model is the correct model, then the Option Implied Volatility σ^I should be exactly the same as the true volatility σ.

Why Options?

Hedging, (speculative) investing, and asset allocation are among the top reasons for option trading.

In essence, options and other derivatives provide a tailored service of risk by slicing, reshaping, and re packaging the existing risks in the underlying security.

The risks are still the same, but investors can choose to take on different aspects of the existing risks in the underlying asset.

Reasons Institutions, Use Equity Derivatives

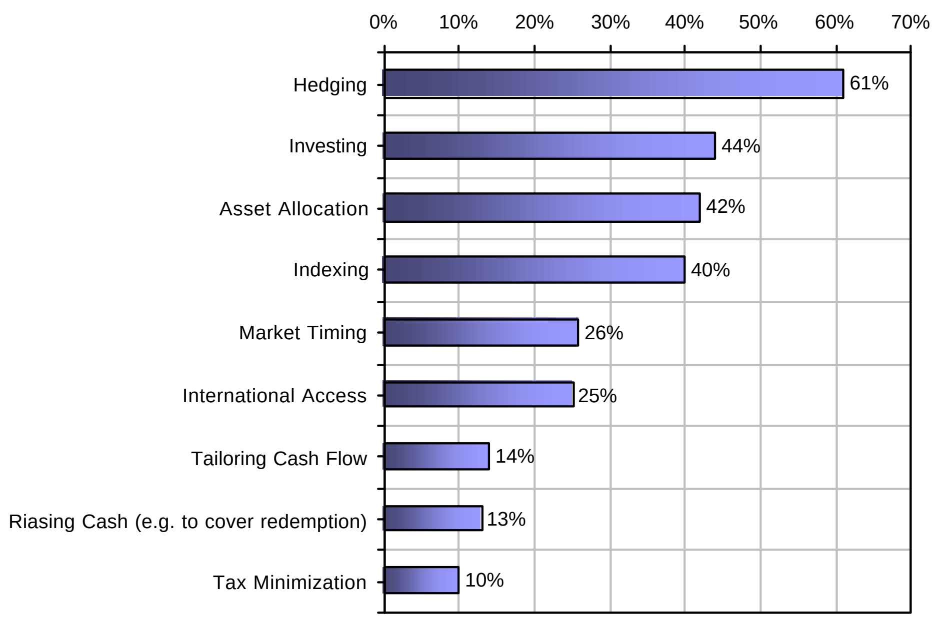

Figure 14: Reasons Institutions, Use Equity Derivatives, Source: Greenwich Associates Survey of 118 Institutional Investors in 1998.

Further Reading

EASY AND ENTERTAINING: "The Universal Financial Device" Chapter 11 of Capital Ideas by Peter Bernstein.

SERIOUS, INTRODUCTORY LEVEL MATERIALS: Options, Futures, and Other Derivative Securities by John Hull.

INSTITUTIONAL, FOR PRACTITIONERS: the Risk magazine, published monthly. http://www.riskpublications.com

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Focus:
BKM Chapters 20.

- p. 652-657, know how to read listed option quotations, difference American vs. European option
- p. 662-670, option strategies
- p. 671-673, put-call parity
- p. 674-679, know basic information about optionlike securities
- p. 683-684, know basic information about exotic options

type of potential questions: concept check question 1, 2, 3 & 4, 6, 8, 9, p. 688, question 14

Questions for the Next Class

Please read:

- BKM Chapters 21, and

- think about the following questions:

There are two sources of uncertainty affecting the SP 500 index:

1. marginal movements due to small chunks of information arrival.
2. market crashes Neither type is diversifiable, and investors are averse to both.

- If we want to gauge the fear of market crash, where do we look?

- Why would someone purchase a deep out of the money put option on the SP 500 index?