15.433 INVESTMENTS

Class 13: The Fixed Income Market

Part 1: Introduction

Spring 2003

Stocks and Bonds

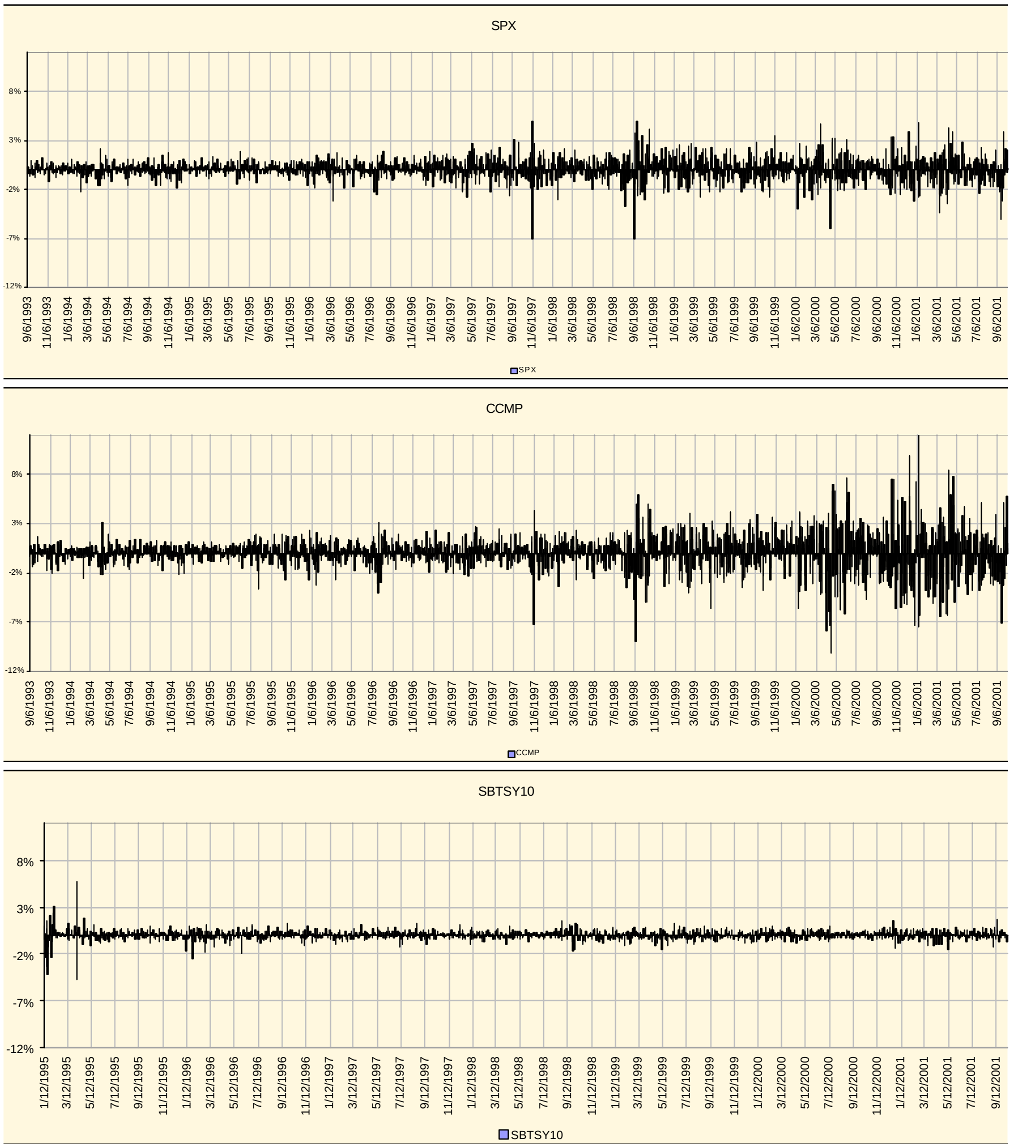

Figure 1: Returns from July 1985 to October 2001 for the S&P 500 index, Nasdaq-index and 10 year Treasury Bonds.

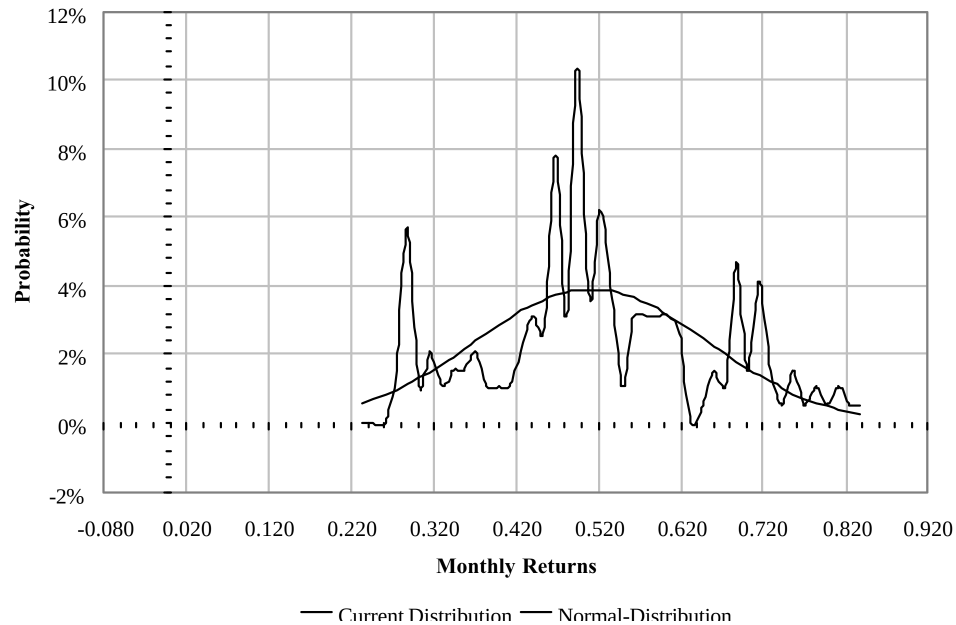

Figure 4: Return-distribution of 1-month Libor rates from 1985 to 2001.

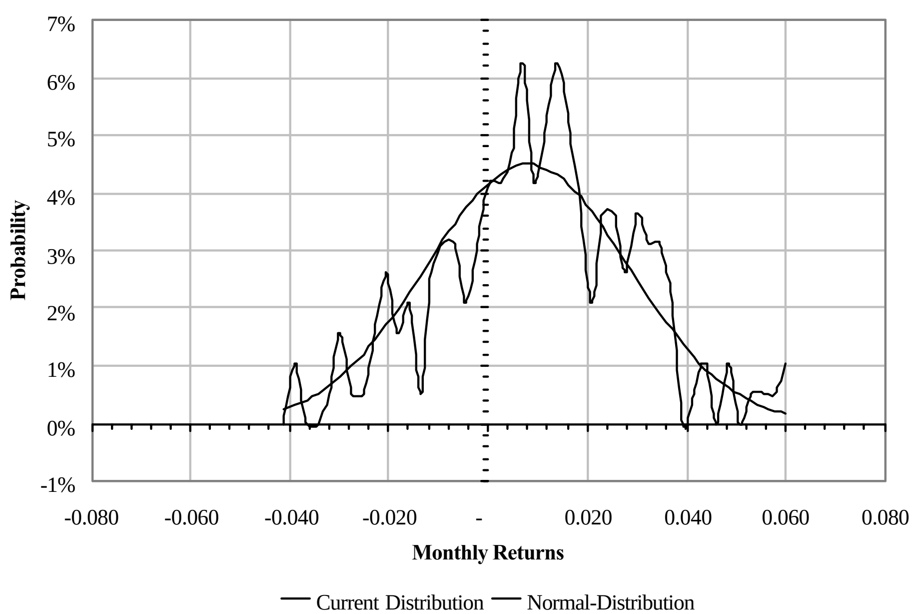

Figure 5: Return-distribution of 10-year US treasury bonds from 1985 to 2001.
Data source for Figures 1, 4, and 5: Bloomberg Professional.

Zero-Coupon Rates

n-year zero $r_{t,t+n}$: the interest rate, determined at time t, of a deposit that starts at time t and lasts for n years.

All the interest and principal is realized at the end of n years. There are no intermediate payments.

Suppose the five-year Treasury zero rate is quoted as 5% per annum. Consider a five-year investment of a dollar:

compounding	\$ 1 grows into
annual	$(1+0.05)^5 = 1.276$
semiannual	$(1+0.05/2)^{10} = 1.280$
continuous	$e^{0.05 \cdot 5} = 1.284$

Zero Coupon Yield-Curve

For any fixed time t, the zero coupon yield curve is a plot of the zero-coupon rate r_{t+n}, with varying maturities n:

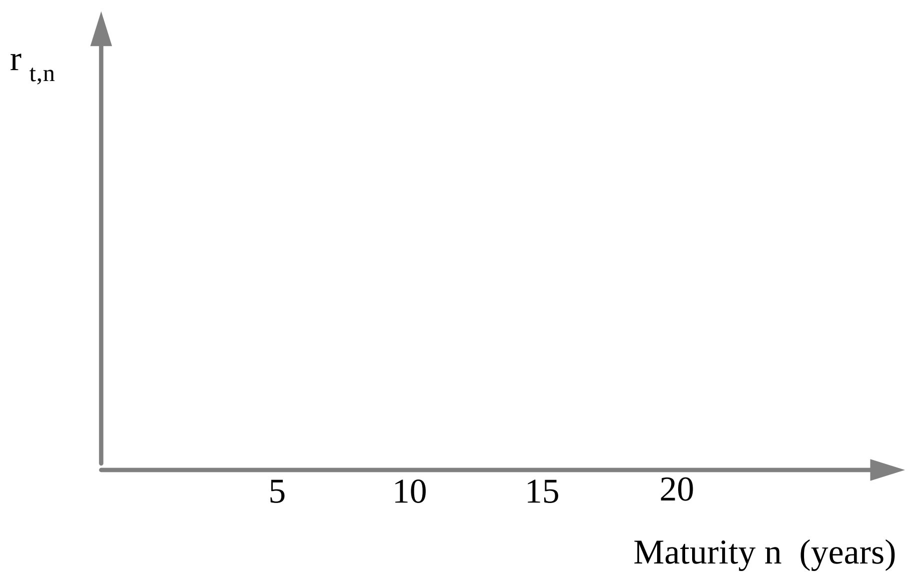

Figure 6: Zero-coupon yield curve.

Treasury Bills

Maturity	to Mat.	Bid	Asked	Chg	Yield
Apr 04 02	3	1.66	1.65	-0.06	1.67
Apr 11 02	10	1.7	1.69	-0.01	1.71
Apr 18 02	17	1.72	1.71	-0.02	1.74
Apr 25 02	24	1.73	1.72	-0.01	1.75
May 02 02	31	1.68	1.67	-0.03	1.7
May 09 02	38	1.69	1.68	-0.03	1.71
May 16 02	45	1.72	1.71	-0.01	1.74
May 23 02	52	1.71	1.7	-0.03	1.73
May 30 02	59	1.71	1.7	-0.03	1.73
Jun 06 02	66	1.72	1.71	-0.04	1.74
Jun 13 02	73	1.74	1.73	-0.03	1.76
Jun 20 02	80	1.74	1.73	-0.03	1.76
Jun 27 02	87	1.75	1.74	-0.03	1.77
Jul 05 02	95	1.76	1.75	-0.03	1.78
Jul 11 02	101	1.76	1.75	-0.04	1.78
Jul 18 02	108	1.78	1.77	-0.04	1.8
Jul 25 02	115	1.8	1.79	-0.03	1.83
Aug 01 02	122	1.85	1.84	....	1.88
Aug 08 02	129	1.88	1.87	....	1.91
Aug 15 02	136	1.91	1.9	....	1.94
Aug 22 02	143	1.91	1.9	....	1.94
Aug 29 02	150	1.92	1.91	+0.01	1.95
Sep 05 02	157	1.96	1.95	....	1.99
Sep 12 02	164	1.99	1.98	....	2.03
Sep 19 02	171	2.02	2.01	....	2.06
Sep 26 02	181	2.06	2.04	....	2.09

T-bills are quoted as bank discount percent r_{BD}. For a \$10'000 par value T-bill sold at P with n days to maturity:

$$r_{BD} = \frac{10'000 - P}{10'000} \cdot \frac{360}{n} \qquad (1)$$

Conversely, the market price of the T-bill is

$$P = 10'000 \cdot \left(1 - r_{BD} \cdot \frac{n}{360}\right) \tag{2}$$

Treasury Bond and Notes

Figure 8: Cash flow representation of a simple bond, Source: $RiskMetrics^{TM}$, p. 109.

Maturity at issue date: T-notes are up to 10 years; T-bonds are from 10 to (30) years.

Coupon payments with rate c%: semiannual (November and May).

Face (par) value: $1,000 or more.

Prices are quoted as a percentage of par value.

If purchased between coupon payments, the buyer must pay, in addition to the quoted (ask) price, accrued interest (the prorated share of the upcoming semiannual coupon).

Some T-bonds are callable, usually during the last five years of the bond's life.

U.S. Government Bonds and Notes

6 5/8	**Apr 02n**	100:12	100:13	-1	1.51
7 1/2	**May 02n**	100:22	100:23	-2	1.53
6 1/2	**May 02n**	100:24	100:25	-2	1.71
6 5/8	**May 02n**	100:25	100:26	-2	1.55
6 1/4	**Jun 02n**	101:03	101:04	-2	1.62
6 3/8	**Jun 02n**	101:04	101:05	-2	1.68
3 5/8	**Jul 02i**	101:20	101:21	....	0.03
6	**Jul 02n**	101:11	101:12	-2	1.80
6 1/4	**Jul 02n**	101:13	101:14	-2	1.86
6 3/8	**Aug 02n**	101:20	101:21	-2	1.92
6 1/8	**Aug 02n**	101:20	101:21	-2	2.03
6 1/4	**Aug 02n**	101:22	101:23	-2	2.04
5 7/8	**Sep 02n**	101:26	101:27	-2	2.10
6	**Sep 02n**	101:28	101:29	-2	2.13
5 3/4	**Oct 02n**	102:00	102:00	-2	2.23
11 5/8	**Nov 02**	105:23	105:24	-3	2.22
5 5/8	**Nov 02n**	102:04	102:05	-1	2.33
5 3/4	**Nov 02n**	102:06	102:07	-2	2.34
5 1/8	**Dec 02n**	101:30	101:31	-2	2.45
5 5/8	**Dec 02n**	102:10	102:11	-2	2.44
4 3/4	**Jan 03n**	101:25	101:26	-2	2.54
5 1/2	**Jan 03n**	102:12	102:13	-2	2.54
6 1/4	**Feb 03n**	103:03	103:04	-2	2.61
10 3/4	**Feb 03**	107:00	107:00	-4	2.59
4 5/8	**Feb 03n**	101:24	101:25	-2	2.62
5 1/2	**Feb 03n**	102:18	102:19	-2	2.60
4 1/4	**Mar 03n**	101:16	101:17	-2	2.67
5 1/2	**Mar 03n**	102:24	102:25	-2	2.66
4	**Apr 03n**	101:09	101:10	-1	2.76
5 3/4	**Apr 03n**	103:04	103:05	-2	2.76
10 3/4	**May 03**	108:19	108:20	-5	2.87
4 1/4	**May 03n**	101:16	101:17	-2	2.89
5 1/2	**May 03n**	102:30	102:31	-2	2.89
3 7/8	**Jun 03n**	101:02	101:03	-2	2.97
5 3/8	**Jun 03n**	102:28	102:29	-3	2.98
3 7/8	**Jul 03n**	101:00	101:01	-2	3.07
5 1/4	**Aug 03n**	102:26	102:27	-3	3.11
5 3/4	**Aug 03n**	103:15	103:16	-2	3.13
11 1/8	**Aug 03**	110:19	110:20	-6	3.15
3 5/8	**Aug 03n**	100:18	100:19	-2	3.18
2 3/4	**Sep 03n**	99:08	99:09	-2	3.23
2 3/4	**Oct 03n**	99:03	99:04	-3	3.31
4 1/4	**Nov 03n**	101:10	101:11	-3	3.38
11 7/8	**Nov 03**	113:08	113:09	-6	3.37

13 1/4	**May 14**	146:02	146:03	-17	5.37
12 1/2	**Aug 14**	142:20	142:21	-18	5.41
11 3/4	**Nov 14**	138:25	138:26	-21	5.46
11 1/4	**Feb 15**	149:00	149:00	-29	5.80
10 5/8	**Aug 15**	143:27	143:28	-34	5.85
9 7/8	**Nov 15**	137:06	137:07	-32	5.87
9 1/4	**Feb 16**	131:14	131:15	-28	5.90
7 1/4	**May 16**	112:12	112:13	-26	5.94
7 1/2	**Nov 16**	114:27	114:28	-27	5.96
8 3/4	**May 17**	127:18	127:19	-28	5.96
8 7/8	**Aug 17**	129:00	129:01	-28	5.96
9 1/8	**May 18**	132:07	132:08	-30	5.98
9	**Nov 18**	131:10	131:11	-30	5.99
8 7/8	**Feb 19**	130:06	130:07	-30	6.00
8 1/8	**Aug 19**	122:15	122:16	-29	6.02
8 1/2	**Feb 20**	126:28	126:29	-30	6.02
8 3/4	**May 20**	129:28	129:29	-30	6.02
8 3/4	**Aug 20**	130:03	130:04	-30	6.02
7 7/8	**Feb 21**	120:17	120:18	-29	6.03
8 1/8	**May 21**	123:17	123:18	-29	6.03
8 1/8	**Aug 21**	123:20	123:21	-30	6.04
8	**Nov 21**	122:14	122:15	-31	6.03
7 1/4	**Aug 22**	114:01	114:02	-29	6.04
7 5/8	**Nov 22**	118:16	118:17	-29	6.04
7 1/8	**Feb 23**	112:21	112:22	-29	6.05
6 1/4	**Aug 23**	102:14	102:15	-26	6.04
7 1/2	**Nov 24**	117:27	117:28	-29	6.04
7 5/8	**Feb 25**	119:14	119:15	-30	6.04
6 7/8	**Aug 25**	110:09	110:10	-28	6.04
6	**Feb 26**	99:15	99:16	-26	6.04
6 3/4	**Aug 26**	108:29	108:30	-29	6.04
6 1/2	**Nov 26**	105:25	105:26	-28	6.04
6 5/8	**Feb 27**	107:13	107:14	-29	6.04
6 3/8	**Aug 27**	104:11	104:12	-27	6.03
6 1/8	**Nov 27**	101:07	101:08	-26	6.03
3 5/8	**Apr 28i**	101:27	101:28	-9	3.51
5 1/2	**Aug 28**	93:06	93:07	-25	6.01
5 1/4	**Nov 28**	90:00	90:00	-24	6.01
5 1/4	**Feb 29**	90:00	90:00	-26	6.00
3 7/8	**Apr 29i**	106:09	106:10	-9	3.51
6 1/8	**Aug 29**	101:27	101:28	-25	5.98
6 1/4	**May 30**	103:29	103:30	-26	5.96
5 3/8	**Feb 31**	93:28	93:29	-24	5.81
3 3/8	**Apr 32i**	99:09	99:10	-8	3.41

Footnote[1]

[1]Footnote: Treasury bond, note and bill quotes are from midafternoon. Colons in bond and note bid-and-asked quotes represent 32nds; 101:01 means 101 1/32. Net change in 32nds. n-Treasury Note. i-Inflation-indexed issue. Treasury bill quotes in hundredths, quoted in terms of a rate of discount. Days to maturity calculated from settlement date. All yields are to maturity and based on the asked quote. For bonds callable prior to maturity, yields are computed to the earliest call date for issues quoted above par and to the maturity date for issues quoted below par.

Bond Pricing with Constant Interest Rate

Assume constant interest rate r with semiannual compounding.

All future cash flows should be discounted using the same interest rate r. (Why?)

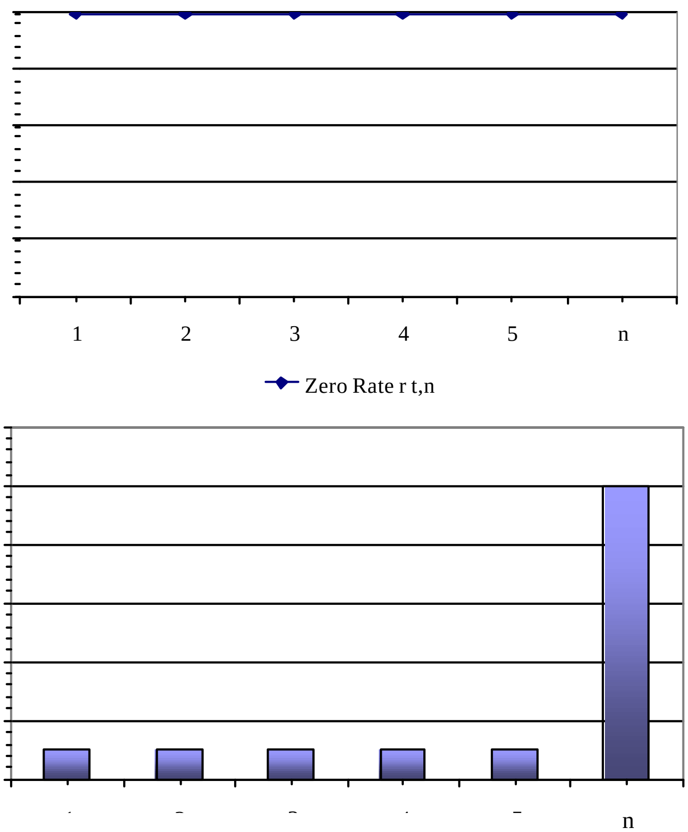

Figure 11: Bond pricing with constant interest rate.

The bond price as a percentage of par value:

$$P = \sum_{i=1}^{10} \frac{\frac{c}{2}}{\left(1+\frac{r}{2}\right)^t} + \frac{100}{\left(1+\frac{r}{2}\right)^t} \tag{3}$$

Time-Varying Interest Rates

In practice, interest rates do not stay constant over time.

If that is the case, then the short- and long-term cash flows could be discounted at different rates. That is, $r_{t,t+n}$ varies over n.

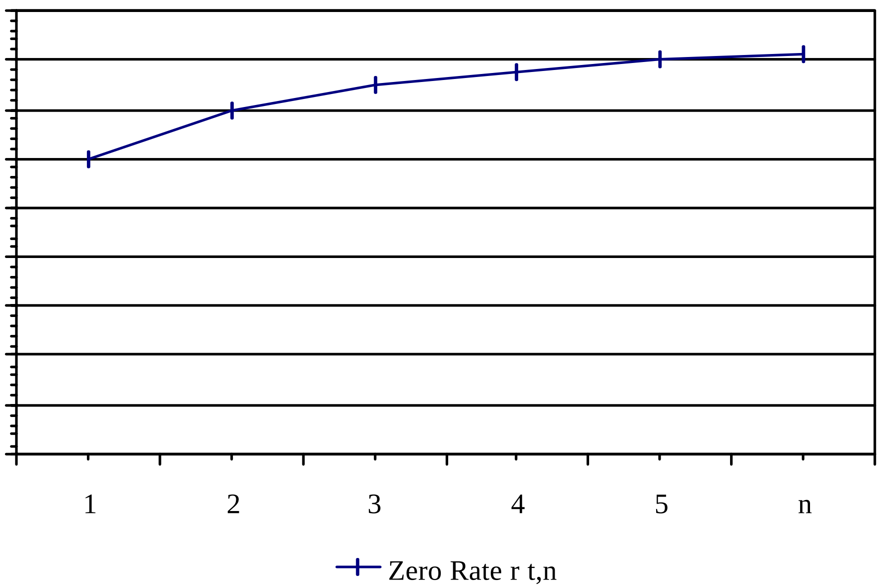

Figure 12: Bond pricing with time varying interest rates.

The time-t bond price as a percentage of par:

$$P = \sum_{i=1}^{10} \frac{\frac{c}{2}}{\left(1 + \frac{r_{0,n_i}}{2}\right)^t} + \frac{100}{\left(1 + \frac{r_{0,n_i}}{2}\right)^{10}} \tag{4}$$

Where $n_1 = 0.5, n_2 = 1, \ldots, n_{10} = 5$

Yield to Maturity

The yield to maturity (YTM) is the interest rate that makes the present value of a bond's payment equal to its price:

$$P = \sum_{i=1}^{10} \frac{\frac{c}{2}}{\left(1 + \frac{YTM}{2}\right)^t} + \frac{100}{\left(1 + \frac{YTM}{2}\right)^{10}} \tag{5}$$

where bond has T-year to maturity and pays semiannual coupon with rate c%:

If the interest rate is a constant r, then the YTM equals r;

In practice, the interest rate is not a constant; The time-t n-year zero-coupon rate $r_{t,t+n}$ varies over time t, and across maturity n;

Intuitively, the YTM for a T-year bond is a weighted average of all zero-coupon rate $r_{t,t+n}$ between $n = 0$ and $n = T$;

What is the difference between the YTM and the holding period return for the same bond?

Duration

$$Duration = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t \tag{6}$$

$$= \frac{1}{P} \sum_{t=1}^{T} \frac{t \cdot (CF_t)}{(1+r)^t} \tag{7}$$

$$= \frac{1}{P} \left[\frac{1 \cdot CF_1}{(1+r)^1} + \frac{2 \cdot CF_2}{(1+r)^2} + \cdots + \frac{T \cdot CF_T}{(1+r)^T} + \right] \tag{8}$$

$$Duration = \frac{\sum_{t=1}^{T} \frac{t \cdot (CF_t)}{(1+r)^t}}{P} = \frac{\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+r)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+r)^t}} \tag{9}$$

$$Modified\ Duration = \frac{Macaulay\ Duration}{1 + \frac{r}{m}} \tag{10}$$

where m represents the number of interest payments per year and r the interest rate.

$$Effective\ Duration = -\frac{1}{B} \cdot \frac{dB}{dy} \tag{11}$$

$$Dollar\ Duration = D \cdot B \tag{12}$$

where B stands for bond value.

Foonote[2]

[2] If nothing else mentioned, we assume that a duration is defined as a modified duration!

Example: An obligation with a redemption price of 100 and an current market-price of 95.27 has a coupon of 6% (annual coupon payments) and has a remaining maturity of 5 years with a yield of 7%. Calculate the Macaulay-Duration.

t	cash flow	pv-factor	pv of cf	cf weight	pv time -weighted with t
#1	#2	#3	#4#=#2·#3	#5=#4/price	#6=#1*#5
1	6.00	0.9346	5.6075	0.05886	0.05886
2	6.00	0.8734	5.2401	0.05501	0.11002
3	6.00	0.8163	4.8978	0.05141	0.15423
4	6.00	0.7629	4.5774	0.04805	0.19219
5	106.00	0.71299	75.5765	0.79329	3.96644
Duration					4.48

Duration Hedge

Recall, that the price change (dP) from a change in yields (dy) is:

$$dP = -D \cdot P \cdot dy \tag{13}$$

[if you have to think twice why D is negative → don't select fixed income portfolio manager as a career option!]

$$\Delta S = -D_S \cdot S \cdot \Delta y \tag{14}$$

$$\Delta F = -D_F \cdot F \cdot \Delta y \tag{15}$$

Where D_S is the duration of the spot position and D_F is the duration of the Futures-position.

$$\sigma_S^2 = (D_S \cdot S)^2 \cdot \sigma_{\Delta y}^2 \tag{16}$$

$$\sigma_F^2 = (D_F \cdot F)^2 \cdot \sigma_{\Delta y}^2 \tag{17}$$

$$\sigma_{SF} = (D_F \cdot F) \cdot (D_S \cdot S) \cdot \sigma_{\Delta y}^2 \tag{18}$$

The number of futures contracts is:

$$N = -\frac{\sigma_{SF}}{\sigma_F^2} = -\frac{D_S \cdot S}{D_F \cdot F} \tag{19}$$

If we have a target duration D_T, we can get it by using:

$$N = -\frac{\sigma_{SF}}{\sigma_F^2} = -\frac{D_S \cdot S}{D_F \cdot F} \tag{20}$$

Example 1: A portfolio manager has a bond portfolio worth \$10 mio. with a modified duration of 6.8 years, to be hedged for 3 months. The current futures price is 93-02, with a notional of \$100'000. We assume that the duration can be measured by CTD, which is 9.2 years.
Compute:

1. The notional of the futures contract;

2. The number of contracts to buy/sell for optimal protection.

Solution:

1. The notional of the futures contract is:

$$(93 + 2/32)/100 \cdot \$100'000 = \$93'062.5 \tag{21}$$

2. The number of contracts to buy/sell for optimal protection.

$$N^* = -\frac{D_S \cdot S}{D_F \cdot F} = -\frac{6.8 \cdot \$10'000'000}{9.2 \cdot \$93'062.5} \tag{22}$$

Note that DVBP of the futures is $9.2 \cdot \$93'062.5 \cdot 0.01\% \;=\; \$85$

Example 2: On February 2, a corporate treasurer wants to hege a July 17 issue of \$ 5 mio. of CP with a maturity of 180 days, leading to anticipated proceeds of \$ 4.52 mio. The September Eurodollar futures trades at 92, and has a notional amount of \$ 1 mio.

Compute:

1. The current dollar value of the of the futures contract;
2. The number of contracts to buy/sell for optimal protection.

Solution:

1. The current dollar value is given by:

$$\$10'000 \cdot (100 - 0.25 \cdot (100 - 92)) = \$980'000 \tag{23}$$

   Note that the duration of futures is 3 months, since this contract refers to 3-month LIBOR.

2. If rates increase, the cost of borrowing will be higher. We need to offset this by a gain, or a short position in the futures. The optimal number of contracts is:

$$N^* = -\frac{D_S \cdot S}{D_F \cdot F} = -\frac{180 \cdot \$4'520'000'000}{90 \cdot \$980'000} = -9.2 \tag{24}$$

   Note that DVBP of the futures is $0.25 \cdot \$1'000'000 \cdot 0.01\% = \$25$

Convexity

The duration should not be used for big swings in the term structure. The accuracy of the estimation of the duration-coefficient depends on the convexity. Duration is an approximate estimate of a convex form with a linear function. The stronger the yield-curve is "curved", the more the real value deviates from the estimated values.

We receive a better estimate applying the first two moments of a Taylor-expansion to estimate the price changes:

$$dP = \frac{dP}{dr} \cdot dr + \frac{1}{2} \cdot \frac{d^2P}{dr^2} dr^2 + \varepsilon \tag{25}$$

ε is the residual part of the Taylor expansion and $\frac{d^2P}{dr^2}$ is the second derivative of the bond price relative to the yield. Dividing both parts by the price we obtain:

$$\frac{dP}{P} = \frac{dP}{dr} \cdot \frac{1}{P} \cdot dr + \frac{1}{2} \frac{d^2P}{dr^2} + \varepsilon \tag{26}$$

Replacing the second derivative of the price equation and reformulating the notation of the Taylor-expansion, we get:

$$\begin{aligned} dP &= \frac{1}{2} \cdot \left[\frac{dP}{dr} \left(-\frac{1}{1+r} \cdot \sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+r_t)^t} \right) \right] \frac{1}{P} \\ &= \frac{1}{2} \cdot \frac{1}{P} \cdot \frac{1}{(1+r)^2} \left[\frac{1 \cdot 2 \cdot CF_1}{(1+r)} + \frac{2 \cdot 3 \cdot CF_2}{(1+r)^2} + \cdots + \frac{T \cdot (T+1) \cdot CF_t}{(1+r)^T} \right] \\ &= \frac{1}{2} \cdot \frac{1}{P} \cdot \frac{1}{(1+r)^2} \cdot \sum_{t=1}^{T} \frac{t \cdot (t+1) \cdot CF_t}{(1+r)^t} \end{aligned} \tag{27}$$

The notation of the price-convexity results from the Taylor-expansion. The first part is the approximation based on the duration. The second part is an approximation based on the convexity of the price/yield-relationship. The percent-approximation results from the duration and the convexity by summing up the individual components.
The convexity is defined as:

$$convexity = \frac{1}{2} \cdot \frac{d^2P}{dr^2} \cdot \frac{1}{P} \tag{28}$$

$$\frac{dP}{P} = \frac{dP}{dr} \cdot \frac{1}{P} \cdot dr + \frac{1}{2} \cdot \frac{d^2P}{dr^2} \cdot \frac{1}{P} (dr)^2 + \varepsilon \tag{29}$$

$$\frac{\Delta P}{P} = -D \cdot \frac{\Delta r}{1+r} + convexity \cdot (\Delta r)^2 \tag{30}$$

Based on duration and convexity the price change for small changes in the market return can be expressed as price change instead a percentage number:

$$\Delta P = -Duration \cdot \frac{\Delta r}{1+r} + convexity \cdot (\Delta r)^2 \tag{31}$$

Building Zero Curves

The coupon-bearing T-notes and bonds can be thought of as packages of zero-coupon securities.

For any time t, a zero-curve builder uses all such coupon-bearing securities traded in the market at time t to calculate the zero-coupon rates $r_{t,t+n}$ for all possible maturities n.

A sophisticated procedure will take into account of the illiquidity and mis-pricing of bonds, as well as tax-related issues.

In recent years coupon-bearing securities have been "stripped" into simpler packages of zero-coupon securities.

For example, a 5-year T-note can be divided up and sold as 10 separate zero-coupon bonds, or as a "strip" of coupon payments and a five-year zero-coupon bond.

Focus:

BKM Chapter 14

- All pages except p. 434 after-tax returns

Style of potential questions: Concept checks 1, 2, 3, 4, 6, 8, 9, p. 443 ff question 1, 5, 14, 15

Preparation for Next Class

Please read:

- BKM Chapter 15, and
- Kao (1993).