15.433 INVESTMENTS

Class 3: Portfolio Theory

Part 1: Setting up the Problem

Spring 2003

A Little History

In March 1952, Harry Markowitz, a 25 year old graduate student from the University of Chicago, published "Portfolio Selection" in the Journal of Finance.

The paper opens with: "The process of selecting a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio".

Thirty eight years later, this paper would earn him a Nobel Prize in economic sciences.

Introduction

Two basic elements of investments:

- the investment opportunity;
- the investor.

Our task for this class:

- a model for financial assets;
- a model for investors;
- optimal portfolio selection.

Modelling Financial Returns

Virtually all real assets are risky. Financial assets, claims on real assets, bear such risk:

- some are designed to minimize risk
- some are designed to capture risk

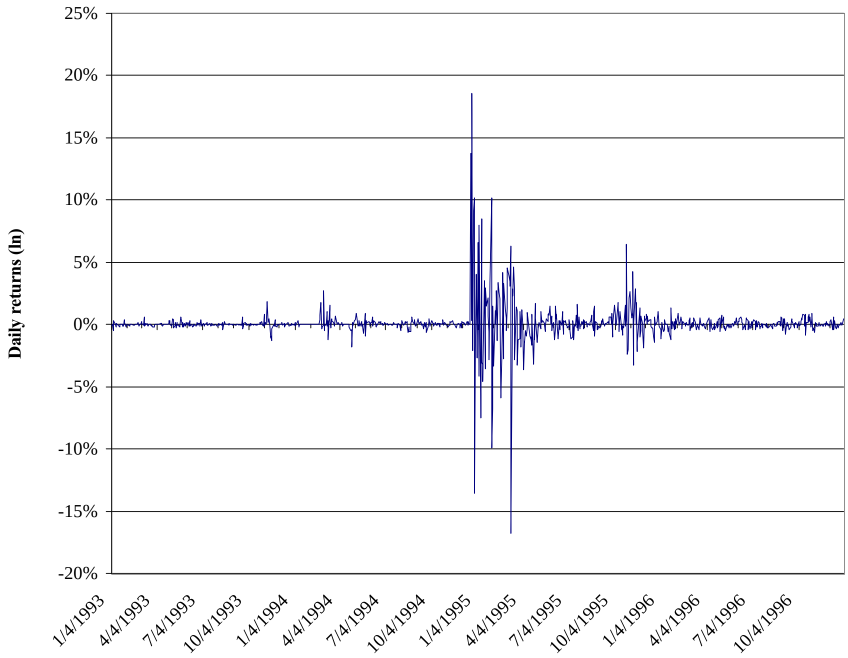

Figure 1: Return of Mexican Peso, Source: Bloomberg Professional.

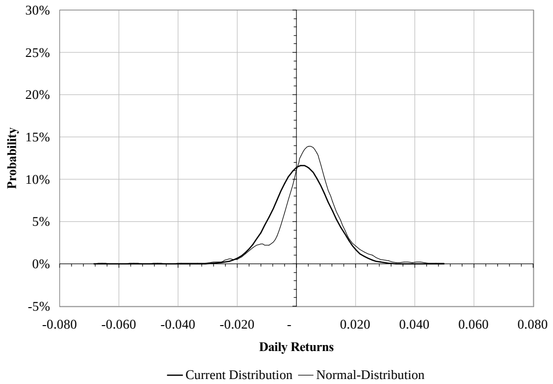

Figure 2: Return of S & P 500 Index, Source: Bloomberg Professional.

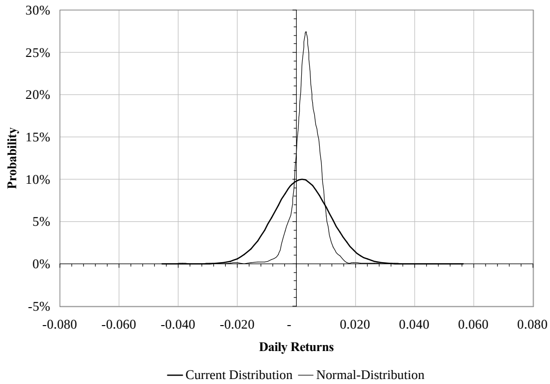

Figure 3: Return of 10 Year Treasury Bills, Source: Bloomberg Professional.

Modelling Investors

Overall, investors are risk averse, although some are more so than the others. *"We next consider the rule that the investor does (or should) consider expected return a desirable thing and variance of return an undesirable thing".* - Markowitz (1952).

Heterogeneity of investors:

- individual investors vs. corporations
- investors with different marginal tax rates
- informed vs. uninformed
- young vs. old
- Behavior issues: loss aversion, mental accounting, over confidence, over reaction, under reaction, etc.

Choose A or B

$$A : \begin{cases} \$240'000 & \text{with probability } 100\% \end{cases}$$

$$B : \begin{cases} \$1'000'000 & \text{with probability } 25\% \\ \$0 & \text{with probability } 75\% \end{cases}$$

Choose C or D

$$C : \begin{cases} -\$750'000 & \text{with probability } 100\% \end{cases}$$

$$D : \begin{cases} \$0 & \text{with probability } 25\% \\ -\$1'000'000 & \text{with probability } 75\% \end{cases}$$

Equivalent Choices:

$$A + D : \begin{cases} \$240'000 & \text{with probability } 25\% \\ -\$760'000 & \text{with probability } 75\% \end{cases}$$

$$B + C : \begin{cases} \$250'000 & \text{with probability } 25\% \\ -\$750'000 & \text{with probability } 75\% \end{cases}$$

Setting up the Problem

What do we need ... a recipe and some ingredients.

The investment opportunity:

- riskfree $r_f = 7\%$
- risky $r_p : E(r_p) = 15\%,\ std(r_p) = 22\%$.

BKM, p. 157 ff.

A mean-variance investor:[1]

$$U(r) = E(r) - 0.005 \cdot A \cdot var(r) \tag{1}$$

The optimal portfolio selection:

- invest a portion y of the total wealth in the risky asset, leaving the rest in the riskfree account
- possible portfolios: $r_y = (1-y) \cdot r_f + y \cdot r_p$
- the optimal portfolio?

  $$\max_{y \in \mathbb{R}} U(r_y)$$

  where R stands for the space of real numbers.

[1] The coefficient 0.005 is in the literature as well written as $\frac{1}{2}$. It is a calibration coefficient to calibrate the subjective risk aversion coefficient A.

Portfolio Construction

The opportunity set is fixed: r_f and r_p

Our only choice variable: y [how much to invest in risk portfolio]

The end product:

$$r_y = (1-y)\cdot r_f + y\cdot r_p \tag{2}$$

$$E(r_y) = E((1-y)\cdot r_f) + E(y\cdot r_p) \tag{3}$$

$$= (1-y)\cdot 0.07 + y\cdot 0.15 \tag{4}$$

$$= 0.07 + 0.08\cdot y \tag{5}$$

$$\begin{aligned} var(r_y) = \; & var((1-y)\cdot r_f) + var(y\cdot r_p) \\ & +2\cdot cov((1-y)\cdot r_f, y\cdot r_p) \end{aligned} \tag{6}$$

$$= 0 + y^2\cdot 0.22^2 + 0 \tag{7}$$

$$= 0.22^2\cdot y^2 \tag{8}$$

$$std(r_y) = \sqrt{var(r_y)} = 0.22|y| \tag{9}$$

The Risk Return Combinations

Every choice of y gives rise to one pair of return E and risk std.

- For y ⩾ 0, we have:

$$y = \frac{E\left(r_y\right) - 0.07}{0.08} = \frac{std\left(r_y\right)}{0.22} \tag{10}$$

- More generally, we have, for any y ¿ 0:

$$y = \frac{E\left(r_y\right) - r_f}{E\left(r_p\right) - r_f} = \frac{std\left(r_y\right)}{std\left(r_p\right)} \tag{11}$$

y may vary over the entire positive real line, but this relation holds regardless.

A linear relation between E and std:

$$E\left(r_y\right) - r_f = \frac{E\left(r_y\right) - r_f}{std\left(r_p\right)} std\left(r_y\right) \tag{12}$$

The Capital Allocation Line

Collecting all $y \in \mathbb{R}$,we get all of the risk-return (μ, σ) combinations available to investors.

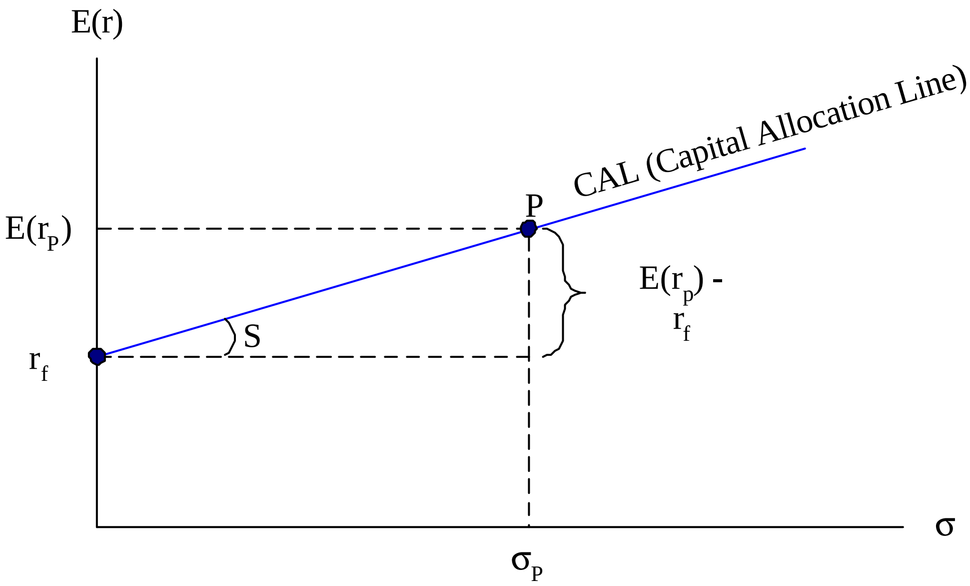

Figure 4: Capital Allocation Line

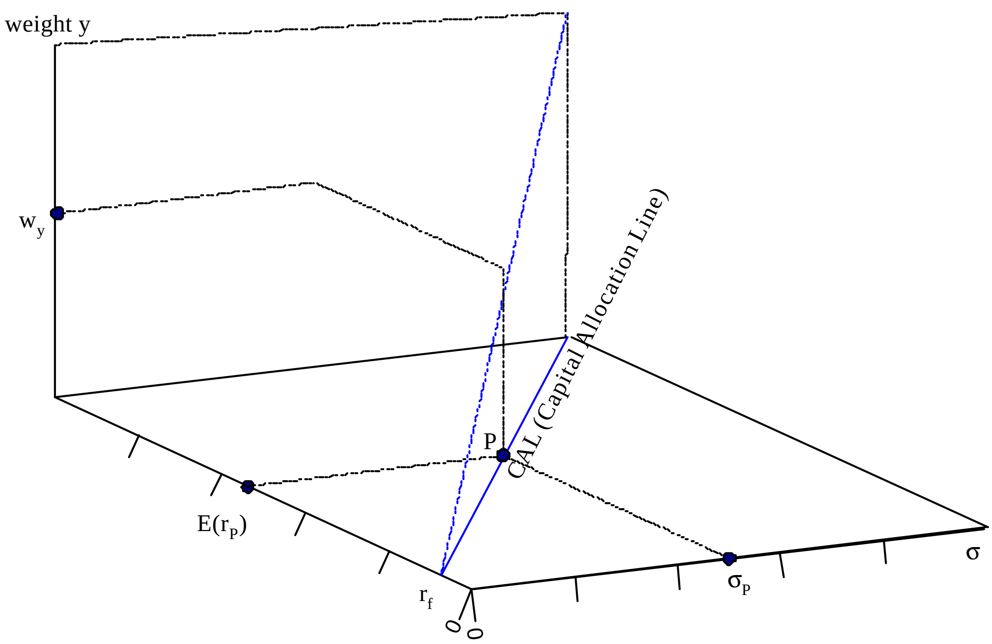

Figure 5: Capital Allocation Line, a different view.

The Sharpe Ratio

One measure of the attractiveness of a portfolio r is its Sharpe Ratio (S):

Intuitively, S measures extra return per extra risk.

$$S = \frac{E(r) - r_f}{std(r)} \tag{13}$$

Recall that the CAL can be re-written as:

$$E(r_y) - r_f = \frac{E(r_y) - r_f}{std(r_p)} std(r_y) \tag{14}$$

For the extra risk std (r_y) chosen (through y), the extra reward is: $S_p \cdot std(r_y)$.

Moreover, the Sharp Ratio S_y of any portfolio thus constructed from r_f and r_p is the same:

$$S_y = S_p = \frac{E(r_y) - r_f}{std(r_p)} \tag{15}$$

for any y $\geqslant$ 0. Does that make sense to you?

Forming the Optimization Problem

We are now ready to "feed" our portfolio to the optimization machine:

$$\begin{matrix} max \\ y \in \mathbb{R} \end{matrix} \; U\left(r_y\right) \tag{16}$$

where

$$U(r) = E(r) - 0.005 \cdot A \cdot var\left(r\right) \tag{17}$$

From our earlier derivation, we know that:

$$E(r_y) = 0.07 + 0.08 \cdot y; \quad var(r_y) = 0.22^2 \cdot y^2 \tag{18}$$

Our optimization problem therefore becomes max f (y):

$$\begin{matrix} max \\ y \in \mathbb{R} \end{matrix} \; U\left(r_y\right) \tag{19}$$

$$f(y) = 0.07 + 0.08 \cdot y - 0.0005 \cdot A \cdot y^2$$

The Optimization Machine

Three components of an optimization problem:

- the objective function $f(y)$;
- the variable y; and
- the search space R

Three ways to solve an optimization problem:

- analytical;
- numerical; and
- graphical.

The mathematical foundation:

- let y* be the solution of $f'(y) = 0$;
- if $f''(y*) < 0$, then y* is truly the optimal solution.

A Pictorial Solution

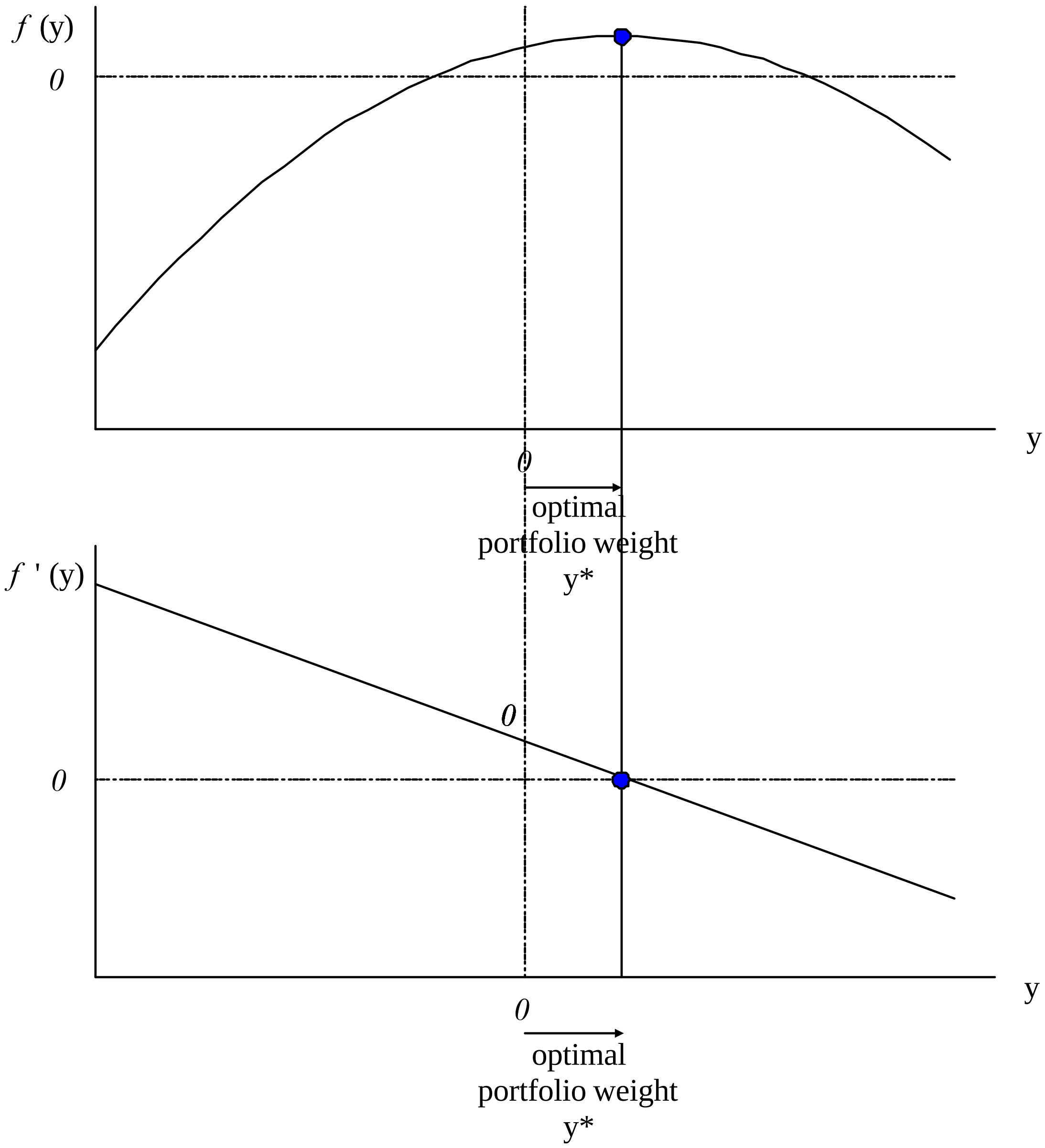

Figure 6: Optimal portfolio weight

An Analytical Solution

The risk aversion coefficient is set at A = 4 and the optimal weight is y*=0.41.

Let's take some derivatives:

$$f'(y) = \frac{\partial f(y)}{\partial y} = 0.08 - 0.22^2 \cdot A \cdot y \tag{20}$$

$$f''(y) = \frac{\partial f'(y)}{\partial y} = -0.22^2 \cdot A \tag{21}$$

1. look for $y*$ that satisfies $f'(y*) = 0$:

$$y* = \frac{0.08}{0.22^2 \cdot A} \tag{22}$$

2. check the optimality of $y*$: $f"(y*) < 0$?

Determinants of Portfolio Weights

More generally, the optimal solution can be expressed as

$$y* = \frac{E\left(r_p - r_f\right)}{var\left(r_p\right) \cdot A} \tag{23}$$

A more risk-averse investor (with a larger risk-aversion coefficient A) will invest less in the risky asset.

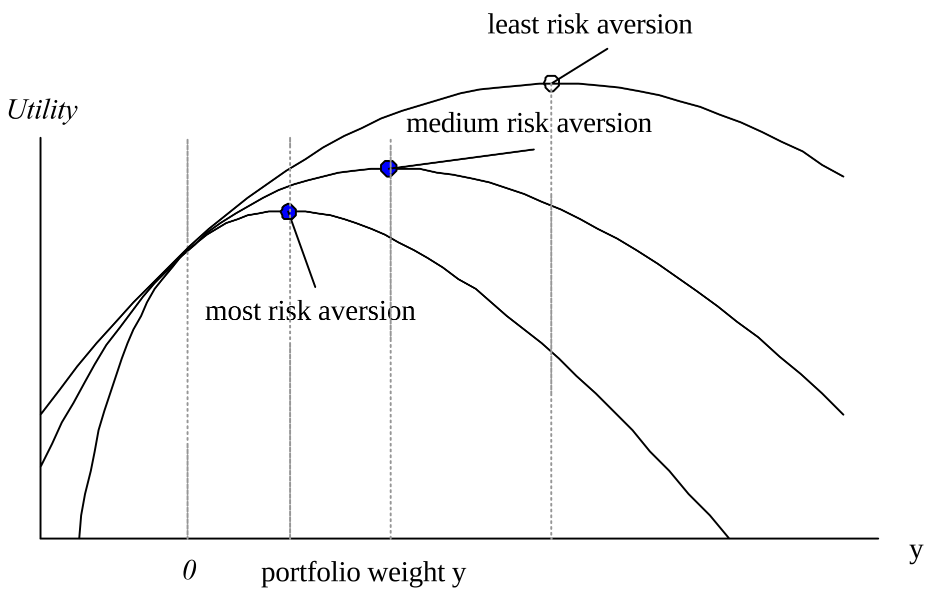

Figure 7: Utility-function.

If the risk premium, $E\left(r_p\right)$ - r_f, of the original risky asset decreases, a risk-averse investor will reduce his holdings in the risky asset accordingly.

If the original asset is risky (with $var(r_p) > 0$), but pays zero risk premium, then no risk-averse investor will hold the risky asset. If the risk premium is negative, a risk-averse investor will start shorting the asset.

What is the optimal portfolio weight y* of an investor with the following utility function?

HINT: This investor cares only about the Sharpe Ratio.

Going Beyond

Our setup assumes the following:

1. A mean variance investor;
2. Investment horizon is fixed to one year;
3. No dynamic rebalancing in between.

Of course, this setup is a very rough characterization of the real investment problem.

Nevertheless, this example is valuable:

- First, it provides a framework for us to think about the portfolio optimization problem.
- Second, although simple, it provides rich intuition.

Now we can go beyond to the next step.

Three Extensions

1. The Skewness Extension: allow skewed asset returns and add preference for positive skewness and aversion to negative skewness.

2. The Horizon Extension: allow investment horizon to vary.

3. The Dynamic Extension: allow for dynamic rebalancing.

Leisure Readings

”Fourteen Pages to Fame,” Chapter 2 of Capital Ideas by Peter Bernstein.

Focus:

Chapters 6 & 7:

- p.157 (eq. 6.1)
- p. 161
- p. 163 to 166
- p. 188
- p. 191 to 195 (utility function, utility curves, CAL)

Reader: Kritzman (1992)

type of potential questions: Chapter 6 concept check question 3 & 4, p. 168 ff. questions 2, 9, 10 chapter 7 concept check question 2, 3, 4 & 5, p. 200 ff. questions 4, 8, 13.

Questions for the Next Class

Please read BKM Appendix A, B of Chapter 6, Black (1995) and Kritzman (1992).

What about market crashes? Can event risks small probability but high impact events ever be ignored in making an investment decision?

What do you think of BKM's defense of mean variance analysis? What is the major assumption in Paul Samuelson's proof? Is this assumption realistic?