M.I.T. **15.450-Spring 2010**
Sloan School of Management **Professor Leonid Kogan**

Handout: Crossing Probabilities of the Brownian Motion

Consider the process X_t, $dX_t = \sigma dZ_t$. Assume that $X_0 = 0$. We want to compute the probability that X_t reaches $a > 0$ before reaching $-b < 0$. Let $f(X_t)$ denote such probability, conditional on the starting value X_t at time t. Note that f does not depend on time explicitly.

By definition of $f(X_t)$ as the probability of hitting the upper boundary first,

$$f(X_t) = \mathrm{E}_t[f(X_{t+dt})]$$

and therefore $f(X)$ satisfies the Kolmogorov backward equation

$$\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial X^2} = 0$$

Note that there are no other terms in this equation, since f does not depend on time explicitly, and X_t has zero drift. The equation $f''(X) = 0$ has a general solution

$$f(X) = A(x + b)$$

Note that in addition to solving the PDE, function $f(X)$ must satisfy the boundary conditions

$$f(a) = 1, \quad f(-b) = 0$$

These conditions follow from the definition of f as the probability of reaching the upper boundary first. With these boundary conditions, we find

$$f(X) = \frac{x + b}{a + b}$$

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