Dynamic Portfolio Choice III

Numerical Approximations in Dynamic Programming

Leonid Kogan

MIT, Sloan

15.450, Fall 2010

Overview

- Approximate the problem with continuous state space using the one with finite state space.
- Finite state space DP problems are easy to implement numerically.
- Many ways to discretize a problem.
- We focus on a particular approach that is general and easy to implement.
- Develop and illustrate the method in the context of a particular application: portfolio optimization with return predictability and margin constraints.

Predictability and Margin Constraints

Problem formulation

- Suppose we observe a price spread between two assets X_t following an AR(1) process

$$X_{t+1} = \rho X_t + \sigma \varepsilon_{t+1}, \quad 0 < \rho < 1, \quad \varepsilon_{t+1} \overset{\text{IID}}{\sim} \mathcal{N}(0, 1)$$

- We would like to design a trading strategy taking advantage of the predictable spread fluctuations.
- Assume that the interest rate is zero.
- A unit trade size generates P&L change of $X_{t+1} - X_t$.
- θ_t is the notional position size at time t.
- The trader starts with W_0 dollars.
- Assume that the margin constraints are such that for every dollar of the absolute trade size, $m > 0$ dollars must be invested in the risk-free asset. Thus, the trade size is constrained by

$$|\theta_t| \leqslant \frac{1}{m} W_t$$

Predictability and Margin Constraints

Problem formulation

- Portfolio value W_t changes according to

$$W_{t+1} = W_t + \theta_t (X_{t+1} - X_t)$$

- Trader maximizes a multi-period objective

$$\mathrm{E}_0 \left[-e^{-\alpha W_T} \right]$$

- If the portfolio value ever becomes negative, the trader is locked out from the market, since the margin constraint

$$|\theta_t| \leqslant \frac{1}{m} W_t$$

excludes further trades.

- We formulate the problem as a dynamic program and solve it numerically.

Predictability and Margin Constraints

DP formulation

- The state vector is

$$Y_t = (W_t, X_t)$$

- Y_t is a controlled Markov process with control θ_t:

$$W_{t+1} = W_t + \theta_t(X_{t+1} - X_t)$$

$$X_{t+1} = \rho X_t + \sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \overset{\text{IID}}{\sim} \mathcal{N}(0, 1)$$

- The Bellman equation takes form

$$J(t, W_t, X_t) = \max_{\theta_t : |\theta_t| \leqslant m^{-1} W_t} \mathrm{E}_t \left[J(t+1, W_{t+1}, X_{t+1}) \right]$$

- We look for the value function $J(t, W_t, X_t)$ satisfying the terminal condition

$$J(T, W_T, X_T) = -e^{-\alpha W_T}$$

Numerical Approximation

Discretizing dynamics

- We want to replace the original problem with a discrete problem amenable to numerical analysis.
- Instead of the original process for the state vector, we introduce a discrete-value controlled Markov chain.
- Replace the spread process X_t with a discrete Markov chain $\widehat{X}_t$ jumping between grid points

$$\widehat{X}(1), \ \widehat{X}(2), \ ..., \widehat{X}(N_X)$$

- Same for the portfolio value process: $\widehat{W}_t$ is a discrete process with values

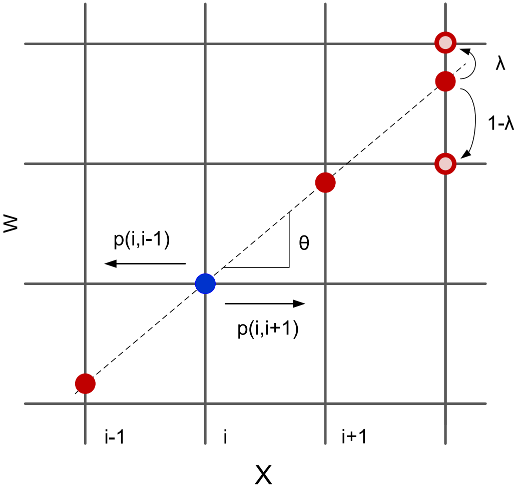
$$\widehat{W}(1), \ \widehat{W}(2), \ ..., \widehat{W}(N_W)$$

- Assume an equally spaced rectangular grid for $\widehat{X}$ and $\widehat{W}$.
- Need to derive transition probabilities on the grid to approximate the distribution of the original state vector.
- Transition probabilities depend on the control θ_t: the discrete process is a controlled Markov chain.

Numerical Approximation

Discretizing dynamics

$\widehat{\mathbf{X}}_t \curvearrowright \widehat{\mathbf{X}}_{t+1}$



Numerical Approximation

Discretizing dynamics

- Consider first the spread process X_t. We want to approximate it with a discrete Markov chain $\widehat{X}_t$ with transition probabilities $p(i, i')$ between grid points i and i':

$$p(i, i') = \text{Prob}(\widehat{X}_{t+1} = \widehat{X}(i') | \widehat{X}_t = \widehat{X}(i))$$

- The transition density of the original process X_t is given by

$$f(X_{t+1}|X_t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_{t+1}-\rho X_t)^2}{2\sigma^2}}$$

- Let $F(X_{t+1}|X_t)$ denote the corresponding CDF.
- Let Δ_X be the spacing of the X-grid.
- We first define the unnormalized transition probabilities for $\widehat{X}_t$ as

$$\widetilde{p}(i, i') = \begin{cases} f\left(\widehat{X}(i')|\widehat{X}(i)\right)\Delta_X, & i' = 2, ..., N_X - 1 \\ F\left(\widehat{X}(1) + \Delta_X/2 \,|\, \widehat{X}(i)\right), & i' = 1 \\ 1 - F\left(\widehat{X}(N_X) - \Delta_X/2 \,|\, \widehat{X}(i)\right), & i' = N_X \end{cases}$$

Numerical Approximation

Discretizing dynamics

- The transition probabilities for $\widehat{X}_t$ are defined as

$$p(i, i') = \frac{\widetilde{p}(i, i')}{\sum_{k=1}^{N_X} \widetilde{p}(i, k)}$$

- To define transition probabilities for $\widehat{W}$, note that for a generic choice of θ,

$$\widetilde{W} = \widehat{W}(j) + \theta(i' - i)\Delta_X$$

  would not be on the W-grid.

- We employ randomization to replace transition to $\widetilde{W}$ with a transition to one of the two points, $\widehat{W}(k)$ or $\widehat{W}(k+1)$, such that $\widehat{W}(k) < \widetilde{W} < \widehat{W}(k+1)$.
- Set the transition probability to $\widehat{W}(k+1)$ equal to

$$p(i, i')\,\lambda, \quad \lambda = \frac{\widetilde{W} - \widehat{W}(k)}{\widehat{W}(k+1) - \widehat{W}(k)}$$

- Note that $\lambda\widehat{W}(k+1) + (1-\lambda)\widehat{W}(k) = \widetilde{W}$.

Numerical Approximation

Discretizing dynamics

- We need to handle the possibility that $\widetilde{W}$ falls outside of the range of the W-grid.
- If $\widetilde{W} > \widehat{W}(N_W)$, we replace $\widetilde{W}$ with $\widehat{W}(N_W)$. This is equivalent to extrapolating the value function to the right of $\widehat{W}(N_W)$ as equal to its value at $\widehat{W}(N_W)$.
- If it happens that $\widetilde{W} < 0$, we set the value function at $(t+1, \widetilde{W}, \widehat{X}_{t+1})$ to

$$-e^{-\alpha \widetilde{W}}$$

  The reason is that the trader is locked out of the market after reaching negative portfolio value levels, and we know the value function following such an event explicitly.

Numerical Approximation

Discrete Bellman equation

- As a result of our discretization approach, we obtain transition probabilities on the grid which depend on the chosen control θ:

$$P\left((i,j),(i',j')|\theta\right)$$

Transition from $(\widehat{X}(i),\widehat{W}(j))$ to $(\widehat{X}(i'),\widehat{W}(j'))$.

- We discretize the possible values of the control (the trade size). Impose the margin constraint so that

$$\frac{\widehat{\theta}_t}{\widehat{W}_t} \in \left\{\widehat{\theta}(1),\ldots,\widehat{\theta}(N_\theta)\right\}, \quad \widehat{\theta}(1) = -\frac{1}{m}, \quad \widehat{\theta}(N_\theta) = \frac{1}{m}$$

- The Bellman equation for the discrete problem takes form

$$\widehat{J}(t,\widehat{W}_t,\widehat{X}_t) = \max_{\widehat{\theta}_t} \mathrm{E}_t\left[\widehat{J}(t+1,\widehat{W}_{t+1},\widehat{X}_{t+1})\right]$$

where the conditional expectation is computed using the transition probabilities $P\left((i,j),(i',j')|\widehat{\theta}_t\right)$.

Numerical Approximation

Parameters

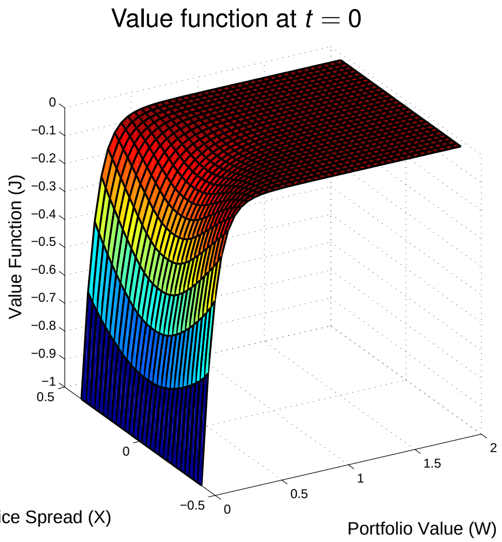
- Assume the following parameters for numerical analysis

α	4
m	0.25
ρ	$\exp(-0.5\Delta t)$
σ	$0.10\sqrt{\Delta t}$
Δt	1/12
T	5

- Time period Δt corresponds to monthly rebalancing of the portfolio.
- Problem horizon T is five years.

Numerical Approximation

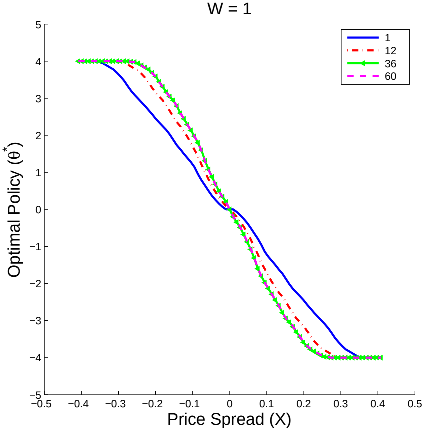
Results



Numerical Approximation

Results

- For a fixed W, plot the (smoothed) optimal portfolio policy as a function of the price spread X at 1, 12, 36, and 60 months left until T.
- Note how the optimal investment strategy depends on the horizon.



15.450 Analytics of Finance
Fall 2010