Review: Arbitrage-Free Pricing and Stochastic Calculus

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Discrete Models

- Definitions of SPD (π) and risk-neutral probability (**Q**).
- Absence of arbitrage is equivalent to existence of the SPD or a risk-neutral probability:

$$P_t = \mathrm{E}_t^{\mathbf{P}}\left[\sum_{u=t+1}^{T} \frac{\pi_u}{\pi_t} D_u\right] = \mathrm{E}_t^{\mathbf{Q}}\left[\sum_{u=t+1}^{T} \frac{B_t}{B_u} D_u\right]$$

- Price of risk: under Gaussian **P** and **Q** distributions,

$$\varepsilon_t^{\mathbf{Q}} = \varepsilon_t^{\mathbf{P}} + \eta_t$$

- Log-normal model (discrete version of Black-Scholes):

$$\mu_t - r_t = \sigma_t \eta_t$$

Problem

- Consider a 3-period model with $t = 0, 1, 2, 3$. There are a stock and a risk-free asset. The initial stock price is \$4 and the stock price doubles with probability $2/3$ and drops to one-half with probability $1/3$ each period. The risk-free rate is $1/4$.
  1. Compute the risk-neutral probability at each node.
  2. Compute the Radon-Nikodym derivative ($d\mathbf{Q}/d\mathbf{P}$) of the risk-neutral measure with respect to the physical measure at each node.
  3. Compute the state-price density at each node.
  4. Compare two assets, both with cash flows only at time 1. One pays $(2, 1)$ in "up" and "down" nodes, the other pays $(3, 0)$. Which one has higher risk premium?

Problem

- A firm is considering a new project. Cash flows form an infinite stream according to the distribution

$$C_t = a + br_t^M + \varepsilon_t$$

  - r_t^M: market returns, IID, $\mathcal{N}(\mu_M, \sigma_M^2)$.
  - ε_t: idiosyncratic shock, IID, $\mathcal{N}(0, \sigma_\varepsilon^2)$.

- Assume that CAPM holds, and the SDF is given by

$$\frac{\pi_{t+1}}{\pi_t} = \exp\left(-r_f - \frac{\eta_M^2}{2} - \eta_M \frac{r_t^M - \mu_M}{\sigma_M}\right)$$

  - $\exp(r_f) - 1$ is the one-period risk-free rate.

1. Compute the present value of cash flows generated by this project.
2. What are the discount factors applied to *expected* cash flows from different periods in the traditional DCF formula?

Stochastic Calculus

- Brownian motion, basic properties (IID Gaussian increments, continuous trajectories, nowhere differentiable).
- Quadratic variation. $[Z]_T = T$. Heuristically,

$$(dZ_t)^2 = dt, \qquad dZ_t\, dt = o(dt)$$

- Stochastic integral: $\int_0^T \sigma_t\, dZ_t$. Basic properties.
- Ito's lemma:

$$df(t, X_t) = \frac{\partial f(t, X_t)}{\partial t}\, dt + \frac{\partial f(t, X_t)}{\partial X_t}\, dX_t + \frac{1}{2}\frac{\partial^2 f(t, X_t)}{\partial X_t^2}\, (dX_t)^2$$

- Multivariate Ito's lemma.

$$df(t, X_t, Y_t) = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial Y_t}\, dY_t + \\ \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\, (dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial Y_t^2}\, (dY_t)^2 + \frac{\partial^2 f}{\partial X_t \partial Y_t}\, dX_t\, dY_t$$

Black-Scholes Model

- Arbitrage-free pricing of options by replication.
- European option with payoff $H(S_T)$.
- Replicating portfolio delta is

$$\theta_t = \frac{\partial f(t, S_t)}{\partial S_t}$$

$$-r\, f(t, S) + \frac{\partial f(t, S)}{\partial t} + rS\frac{\partial f(t, S)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f(t, S)}{\partial S^2} = 0$$

with the boundary condition $f(T, S) = H(S)$.

- Derive the B.-S. PDE using replication arguments:

$$df(t, S_t) = \theta_t dS_t + (f(t, S_t) - \theta_t S_t)r\, dt$$

Problem

- Your colleagues have developed a term structure model that they intend to use for pricing of interest-rate sensitive securities. Their model is of the following form: they fit the shape of the term structure using a parsimonious closed-form description, and then describe the evolution of necessary parameters. For example, one specification of the bond yields y^τ is

$$y_t^\tau = a + \frac{1}{b+\tau} x_t$$
$$dx_t = -\theta(x_t - \overline{x})\,dt + \sigma dZ_t$$

You suspect that this model implies arbitrage opportunities. How can you convince your colleagues that this is the case?

Solution

- We want to show inconsistencies in returns on zero-coupon bonds of different maturities that lead to arbitrage.
- Consider prices of bonds maturing at different dates. For maturity T,

$$P_t^T = \exp[-y_t^{T-t}(T-t)] = \exp\left[-a(T-t) - \frac{(T-t)}{b+(T-t)}x_t\right]$$

- Compute bond returns. Let $\tau = T - t$. Using Ito's formula,

$$\frac{dP_t^T}{P_t^T} = \left(a + \frac{b}{(b+\tau)^2}x_t + \frac{\tau}{b+\tau}\theta(x_t - \overline{x}) + \frac{1}{2}\left[\frac{\tau}{b+\tau}\right]^2\sigma^2\right)dt - \frac{\tau}{b+\tau}\sigma dZ_t$$

- To avoid arbitrage, expected excess returns on bonds of different maturity have to satisfy a single-factor pricing relation:

$$\text{Risk Premium}(\tau) = \lambda_t \sigma_t^\tau$$

  where σ_t^τ is the diffusion coefficient of bond returns with maturity τ.
- Interest rate is $r_t = y_t^0 = a + b^{-1}x_t$, so our computation above yields

$$\text{Risk Premium}(\tau) = \left[\frac{b}{(b+\tau)^2} - \frac{1}{b}\right]x_t + \frac{\tau}{b+\tau}\theta(x_t - \overline{x}) + \frac{1}{2}\left[\frac{\tau}{b+\tau}\right]^2\sigma^2$$

  Risk premia implied by the model do not have a one-factor structure, and therefore one can construct an explicit arbitrage trade.

Solution

- Consider two bonds with risk premia Risk $\text{Premium}_{i,t}$, $i = 1, 2$ and diffusion coefficients of returns $\sigma_{i,t}$. Assume that the risk premia do not have a one-factor structure, and therefore we can find two bonds such that

$$\frac{\text{Risk Premium}_{1,t}}{\sigma_{1,t}} > \frac{\text{Risk Premium}_{2,t}}{\sigma_{2,t}}$$

- Construct a portfolio with \$1 total value, $\sigma_{1,t}^{-1}$ dollars in bond 1, $-\sigma_{2,t}^{-1}$ dollars in bond 2 and the rest in the short-term risk-free asset. The risk premium on this portfolio is

$$\frac{\text{Risk Premium}_{1,t}}{\sigma_{1,t}} - \frac{\text{Risk Premium}_{2,t}}{\sigma_{2,t}} > 0$$

  This is arbitrage, since the portfolio has no risk, and such risk-free excess returns can be generated at all times.

Pricing by Replication: Limitations

- In many models one cannot derive a unique price for a derivative.
- Term structure models, stochastic volatility.
- Price assets relative to each other. Replication argument combined with assumptions on prices of risk.
- Alternatively, specify dynamics directly under **Q**.

Risk-Neutral Pricing

- General pricing formula

$$P_t = \mathrm{E}_t^{\mathbf{Q}} \left[\exp\left(-\int_t^T r_s \, ds \right) H_T \right]$$

- Need to specify dynamics of the underlying under **Q**.
- If underlying is a stock, only one way to do this: set expected return to r.
- **Q** dynamics is related to **P** through price of risk

$$dZ_t^{\mathbf{P}} = -\eta_t \, dt + dZ_t^{\mathbf{Q}}$$

- Risk premium

$$\mathrm{E}_t^{\mathbf{P}} \left[\frac{dS_t}{S_t} \right] - r_t \, dt = \mathrm{E}_t^{\mathbf{P}} \left[\frac{dS_t}{S_t} \right] - \mathrm{E}_t^{\mathbf{Q}} \left[\frac{dS_t}{S_t} \right]$$

Problem

- Suppose that uncertainty in the model is described by two independent Brownian motions, $Z_{1,t}$ and $Z_{2,t}$. Assume that there exists one risky asset, paying no dividends, following the process

$$\frac{dS_t}{S_t} = \mu(X_t)\, dt + \sigma\, dZ_{1,t}$$

where

$$dX_t = -\theta X_t\, dt + dZ_{2,t}$$

The risk-free interest rate is constant at r.

1. What is the price of risk of the Brownian motion $Z_{1,t}$?
2. Give an example of a valid SPD in this model.
3. Suppose that the price of risk of the second Brownian motion, $Z_{2,t}$, is zero. Characterize the SPD in this model.
4. Derive the price of a European Call option on the risky asset in this model, with maturity T and strike price K.

Risk-Neutral Pricing and PDEs

- Derive a PDE on derivative prices using Ito's lemma.
- One-factor term structure model

$$\mathrm{E}_t^{\mathbf{Q}}[df(t, r_t)] = r_t f(t, r_t)\, dt$$

- Vasicek model:

$$dr_t = -\kappa(r_t - \bar{r})\, dt + \sigma\, dZ_t^{\mathbf{Q}}$$

- $f(t, r_t)$ must satisfy the PDE

$$\frac{\partial f(t, r)}{\partial t} - \kappa(r - \bar{r})\frac{\partial f(t, r)}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 f(t, r)}{\partial r^2} = rf(t, r)$$

  with the boundary condition

$$f(T, r) = 1$$

- Expected bond returns satisfy

$$\mathrm{E}_t^{\mathbf{P}}\left(\frac{dP(t, T)}{P(t, T)}\right) = (r_t + \sigma_t^P \eta_t)\, dt$$

Problem

- Suppose that, under **P**, the price of a stock paying no dividends follows

$$\frac{dS_t}{S_t} = \mu(S_t)\,dt + \sigma(S_t)\,dZ_t$$

Assume that the SPD in this market satisfies

$$\frac{d\pi_t}{\pi_t} = -r\,dt - \eta_t dZ_t$$

1. How does η_t relate to r, μ_t, and σ_t?
2. Suppose that there exists a derivative asset with price $C(t, S_t)$. Derive the instantaneous expected return on this derivative as a function of t and S_t.
3. Derive the PDE on the price of the derivative $C(t, S)$, assuming that its payoff is given by $H(S_T)$ at time T.
4. Suppose that there is another derivative trading, with a price $D(t, S_t)$ which does not satisfy the PDE you have derived above. Construct a trading strategy generating arbitrage profits using this derivative, the risk-free asset and the stock.

Monte Carlo Simulation

- Random number generation: inverse transform, acceptance-rejection method.
- Variance reduction: antithetic variates, control variates.
- Intuition behind control variates: carve out the part of the estimated moment that is known in closed form, no need to estimate that by Monte Carlo.
- Good control variates: highly correlated with the variable of interest, expectation known in closed form.
- Examples of control variates: stock price, payoff of similar option, etc.

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