15.997 Practice of Finance: Advanced Corporate Risk Management
Spring 2009

Pricing Risk

MIT Sloan School of Management
15. 997 Advanced Corporate Risk Management
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Overview

- Two Alternative Methods for Discounting Cash Flows
- Risk-Neutral Pricing – an Introduction
  - State Prices
  - The Risk-Neutral Probability Distribution
  - The Forward Price as a Certainty-Equivalent
- Implementing Risk-Neutral Valuation
  - The Risk-Neutral Distribution in Binomial Trees
    - Random Walk Example
    - Mean Reversion Example
  - Valuation Mechanics – when risk-adjusted works
    - Example #1: Single-Period, Symmetric Risk
    - Example #3: Two-Period, Compounded Risk
  - Valuation Mechanics – when risk adjusted doesn't work
    - Example #2: Single-Period, Skewed Risk
    - Example #4: Two-Period, Non-compounded Risk
- Turbocharged Valuation
- Equilibrium Risk Pricing, Arbitrage Pricing & Risk-Neutral Valuation

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Two Alternative Methods for Discounting Cash Flows

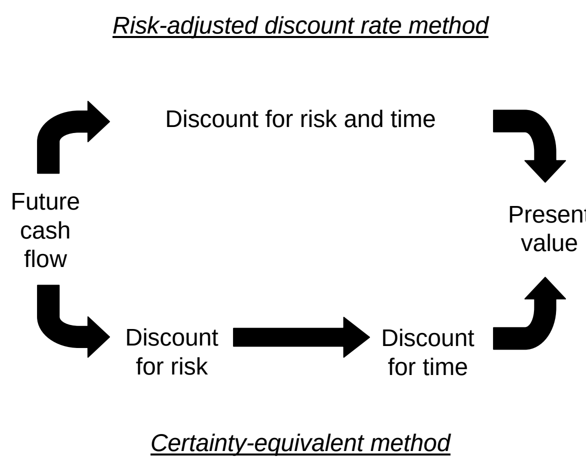

Figure 9.1 from *Lecture Notes on Advanced Corporate Financial Risk Management.* Used with permission.

Illustration of the Two Methods

Table 7.1
Hejira Corporation
Two Alternative Methods for Valuing Oil Production @ Forecasted Spot Prices

Method #1: Risk Adjusted Discount Rate Method -- simultaneously adjust for risk and time					
Year	1	2	3	4	5
Forecasted Production (000 bbls)	10,000	9,000	8,000	7,000	6,000
Forecasted Spot Price ($/bbl) -- current price $38	35.00	33.50	32.75	32.38	32.19
Forecasted Spot Revenue ($ 000)	350,000	301,500	262,000	226,625	193,125
Risk-adjusted Discount Rate, r_a	10.0%	10.0%	10.0%	10.0%	10.0%
Risk-adjusted Discount Factor	0.9048	0.8187	0.7408	0.6703	0.6065
PV ($ 000)	316,693	246,847	194,094	151,911	117,136
Total PV Spot Sales ($ 000)	1,026,682				
Method #2: Certainty Equivalent Method -- separately adjust for risk then for time					
Forecasted Spot Revenue ($ 000)	350,000	301,500	262,000	226,625	193,125
Certainty Equivalence Risk Premium, λ	6.0%	6.0%	6.0%	6.0%	6.0%
Certainty Equivalence Factor	94.2%	88.7%	83.5%	78.7%	74.1%
Certainty Equivalent Revenue	329,618	267,407	218,841	178,270	143,071
Riskless Discount Rate, r_f	4.0%	4.0%	4.0%	4.0%	4.0%
Riskless Discount Factor	0.9608	0.9231	0.8869	0.8521	0.8187
PV ($ 000)	316,693	246,847	194,094	151,911	117,136
Total PV Spot Sales ($ 000)	1,026,682				

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The Risk-Premium

- The risk-adjusted discount rate equals the risk-free rate plus a risk-premium:

$$r_a = r_f + \lambda$$

- Where does this risk-premium come from?
- How general is this risk-premium? If I have a risk-premium for oil projects, can I apply it to all oil projects?
- The risk-neutral methodology is one incarnation of the certainty-equivalent method, where we drill down beyond the risk-premium, λ, and then implement the certainty-equivalent calculations in a specific way.

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State Prices

The General Idea

- Start with a project that has cash flows tied to an underlying risk factor, S. Suppose we can successfully value that project by discounting the cash flows in the usual manner using a risk-premium, λ.
- Unfortunately, we <u>cannot</u> use the same risk-premium, λ, to value all other projects with cash flows tied to S! As the structure of the dependency on S changes, the risk-premium has to change. But how?
- We need to dig deeper. It turns out that we are ignoring some information that we already have.
- The risk structure of S – is it a random walk with a given drift and standard deviation, or is it mean reverting, or something else – is the information that we are ignoring. We can combine information about this risk structure together with the knowledge of the risk-premium, λ, for any single project to recover the extra information that allows us to value all other projects with a dependence on S.
- What is the extra information? It is a complete vector of state-contingent prices for risk, φ. With this vector in hand, we can easily value all projects dependent on S.
- There is no contradiction between risk-adjusted discounting using a risk-premium, λ, and risk-neutral pricing using state prices. They are tied together and, used correctly, do the same thing. But risk-neutral pricing is the more convenient and robust route for handling many different types of projects with different contingencies and changing risk profiles.

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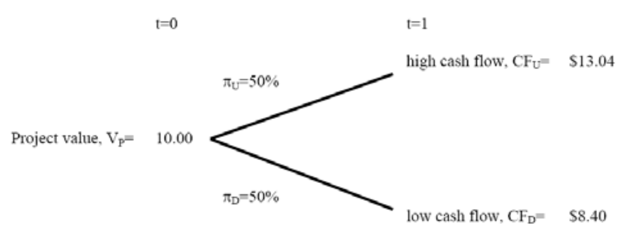
The Starting Point

Figure 9.2
Project Cash Flows



The binomial branching shown is for an <u>underlying</u> risk factor *S*. The risk factor starts in *t*=0 at the value S_0, and branches in *t*=1 to either S_U or S_D. This figure shows the branching of *S*, but not those values. Instead, we <u>overlay</u> onto the branching diagram the characteristics of a project with cash flows contingent on *S*. At *t*=1 we show those contingent cash flows. At *t*=0 we show the project value.

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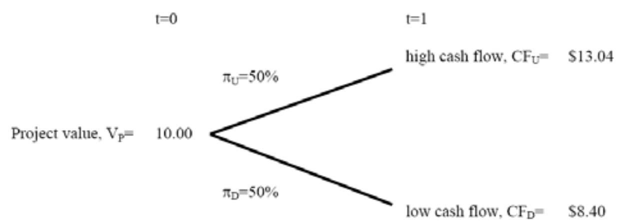
The Starting Point (cont.)

Figure 9.2
Project Cash Flows



Assuming r_f = 4%, the fact that V_P = $10 implies r_a = 7% so that λ = 3%.

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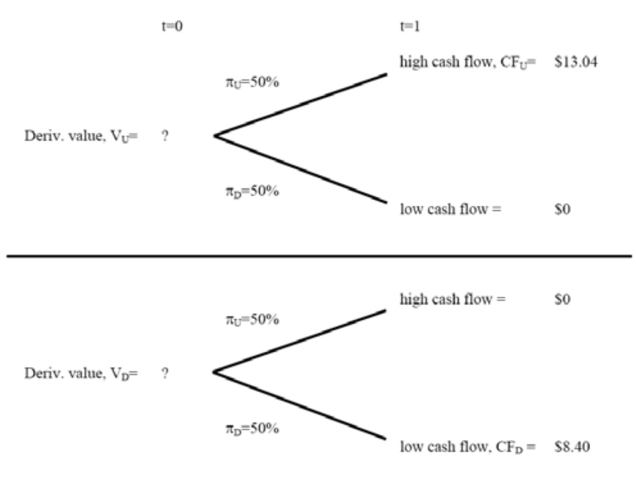
The Complication: How Do We Handle Other Projects Also Contingent on *S*?

Figure 9.3
Derivative Cash Flows



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Applying the Same Risk-premium is Wrong

- It is tempting to apply the same risk-adjusted discount rate or risk-premium to the expected cash flows for each of these derivative projects. But this would be incorrect:

$$E[CF]e^{-r_a} = (CF_U \times \pi_U)e^{-r_a} = (CF_U \times \pi_U)e^{-\lambda_P}\ e^{-r_f} \neq V_U$$

$$E[CF]e^{-r_a} = (CF_D \times \pi_D)e^{-r_a} = (CF_D \times \pi_D)e^{-\lambda_P}\ e^{-r_f} \neq V_D$$

- How do we know this?

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Revalue the Project Using “Forward State Prices”

- Rewrite the valuation of the project. But instead of discounting the whole expected cash flow by the risk-premium, λ, discount the two separate contingent cash flows, CF_U and CF_D, by two different discount factors, φ_U and φ_D:

$$V_P = (CF_U \times \pi_U \times \phi_U + CF_D \times \pi_D \times \phi_D)e^{-r_f}$$

- If this were a single equation, there would be no sense in trying to separate the two cash flows and applying two different discount factors. But, in fact, we can combine that one equation together with the equation for valuing a risk-free cash flow:

$$B = (\$1 \times \pi_U \times \phi_U + \$1 \times \pi_D \times \phi_D)e^{-r_f}$$

- This gives us two equations in two unknowns, so we can solve for the two different discount factors, φ_U and φ_D.

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The Forward State Prices or State Contingent Discount Factors

- Solving these two equations gives us:

$$\varphi_U = \left(\frac{V\,e^{r_f} - CF_D}{CF_U - CF_D} \right) \frac{1}{\pi_U} = 0.866$$

$$\varphi_D = \frac{1 - \pi_U\,\varphi_U}{1 - \pi_U} = 1.134$$

- Note that the discount factor being applied to "up" cash flows is less than 1, while the discount factor being applied to "down" cash flows is greater than 1.
- Cash in the "down" state is like insurance, and we are willing to pay a premium for it! Makes sense.

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Valuing the Derivatives with these Forward State Prices

$$V_U = \left(CF_U \times \pi_U \times \phi_U \right) e^{-r_f} = \$5.42$$

$$V_D = \left(CF_U \times \pi_U \times \phi_U + CF_D \times \pi_D \times \phi_D \right) e^{-r_f} = \$4.58$$

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Valuing the Derivatives with these Forward State Prices

$$V_U = \left(CF_U \times \pi_U \times \phi_U\right) e^{-r_f} = \$5.42$$

$$V_D = \left(CF_U \times \pi_U \times \phi_U + CF_D \times \pi_D \times \phi_D\right) e^{-r_f} = \$4.58$$

- Now, if we want to, we can back out the risk-premium that works in a classical risk-adjusted discount calculation:

$$\lambda_U \text{ solves } V_U = \left(CF_U \times \pi_U\right) e^{-\lambda_U} e^{-r_f} \text{ which implies } \lambda_U = 14.4\%$$

$$\lambda_D \text{ solves } V_D = \left(CF_D \times \pi_D\right) e^{-\lambda_D} e^{-r_f} \text{ which implies } \lambda_D = -12.6\%$$

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Forward State Prices vs. a Single Risk-Premium

- The forward state prices are discount factors that are specific to the particular branch or state on the tree of the underlying risk factor, *S*.
- The forward state prices can be used to value <u>any</u> project with cash flows contingent on *S*.
- The project risk-premium, λ_P, reflects the particular weighting of “up” and “down” cash flows. It <u>cannot</u> be used to value any project with cash flows contingent on *S*. It only makes sense for projects with a similar weighting of “up” and “down” cash flows.
- But we can use the project risk-premium, λ_P, together with knowledge about the risk structure of S, to derive the forward state prices.

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The Risk-Neutral Distribution

Indeterminacy

- In the previous section on state prices, all of our illustrations assumed a known probability for the “up” branching and the “down” branching.
- We then backed out the forward state prices, and then valued the derivatives.
- What if we had assumed a different pair of probabilities for the “up” branching and the “down” branching?
  - The forward state prices would have changed.
  - But the derivative values would NOT have changed!
- Why?
  - Because everywhere in the valuation formulas we employ the product of the forward state prices and the probability. This is also true in the equations with which we derive the forward state prices.
  - When we change the probability, we change the forward state prices, but the product of the probability and the state prices stays the same.
  - Therefore all valuations remain the same.

Illustration: Start with the Original Problem

Table 9.2
Derivative Valuation for an Alternative Probability Assumption

	Original
Inputs	
CF_U	\$13.04
CF_D	\$8.40
V_P	\$10.00
π_U	50%
π_D	50%
r_f	4.0%
Outputs	
φ_U	0.866
φ_D	1.134
$\pi^*_U=\pi_U\varphi_U$	0.433
$\pi^*_D=\pi_D\varphi_D$	0.567
V_U	\$5.42
V_D	\$4.58
V_{LP}	\$5.00
λ_P	3.0%
λ_U	14.4%
λ_D	-12.6%
λ_{LP}	5.8%

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Illustration: Revise it by substituting new probabilities

Table 9.2
Derivative Valuation for an Alternative Probability Assumption

	Original	Revised
Inputs		
CF_U	\$13.04	
CF_D	\$8.40	
V_P	\$10.00	
π_U	50%	60%
π_D	50%	40%
r_f	4.0%	
Outputs		
φ_U	0.866	
φ_D	1.134	
$\pi^*_U=\pi_U\varphi_U$	0.433	
$\pi^*_D=\pi_D\varphi_D$	0.567	
V_U	\$5.42	
V_D	\$4.58	
V_{LP}	\$5.00	
λ_P	3.0%	
λ_U	14.4%	
λ_D	-12.6%	
λ_{LP}	5.8%	

← change

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Illustration: Revise it by substituting new probabilities

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φ_U	0.866	
φ_D	1.134	
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$\pi^*_D=\pi_D\varphi_D$	0.567	
V_U	\$5.42	
V_D	\$4.58	
V_{LP}	\$5.00	
λ_P	3.0%	
λ_U	14.4%	
λ_D	-12.6%	
λ_{LP}	5.8%	

constant

change

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Illustration: Recalculate the outputs

Table 9.2
Derivative Valuation for an Alternative Probability Assumption

	Original	Revised
Inputs		
CF_U	\$13.04	
CF_D	\$8.40	
V_P	\$10.00	
π_U	50%	60%
π_D	50%	40%
r_f	4.0%	
Outputs		
φ_U	0.866	0.721
φ_D	1.134	1.418
$\pi^*_U=\pi_U\varphi_U$	0.433	0.433
$\pi^*_D=\pi_D\varphi_D$	0.567	0.567
V_U	\$5.42	\$5.42
V_D	\$4.58	\$4.58
V_{LP}	\$5.00	\$5.00
λ_P	3.0%	7.2%
λ_U	14.4%	32.7%
λ_D	-12.6%	-34.9%
λ_{LP}	5.8%	13.9%

changed

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Illustration: Recalculate the outputs

Table 9.2
Derivative Valuation for an Alternative Probability Assumption

	Original	Revised
Inputs		
CF_U	\$13.04	
CF_D	\$8.40	
V_P	\$10.00	
π_U	50%	$\boxed{60\%}$
π_D	50%	$\boxed{40\%}$
r_f	4.0%	
Outputs		
φ_U	0.866	0.721
φ_D	1.134	1.418
$\pi^*_U=\pi_U\varphi_U$	0.433	0.433
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V_U	\$5.42	\$5.42
V_D	\$4.58	\$4.58
V_{LP}	\$5.00	\$5.00
λ_P	3.0%	7.2%
λ_U	14.4%	32.7%
λ_D	-12.6%	-34.9%
λ_{LP}	5.8%	13.9%

changed

constant

23

Illustration: Recalculate the outputs

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Derivative Valuation for an Alternative Probability Assumption

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Inputs		
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Outputs		
φ_U	0.866	0.721
φ_D	1.134	1.418
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$\pi^*_D=\pi_D\varphi_D$	0.567	0.567
V_U	\$5.42	\$5.42
V_D	\$4.58	\$4.58
V_{LP}	\$5.00	\$5.00
λ_P	3.0%	7.2%
λ_U	14.4%	32.7%
λ_D	-12.6%	-34.9%
λ_{LP}	5.8%	13.9%

changed

constant

constant

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Illustration: Recalculate the outputs

Table 9.2
Derivative Valuation for an Alternative Probability Assumption

	Original	Revised	
Inputs			
CF_U	\$13.04		
CF_D	\$8.40		
V_P	\$10.00		
π_U	50%	$\boxed{60\%}$	
π_D	50%	$\boxed{40\%}$	
r_f	4.0%		
Outputs			
φ_U	0.866	0.721	changed
φ_D	1.134	1.418	
$\pi^*_U = \pi_U \varphi_U$	0.433	0.433	constant
$\pi^*_D = \pi_D \varphi_D$	0.567	0.567	
V_U	\$5.42	\$5.42	constant
V_D	\$4.58	\$4.58	
V_{LP}	\$5.00	\$5.00	
λ_P	3.0%	7.2%	changed
λ_U	14.4%	32.7%	
λ_D	-12.6%	-34.9%	
λ_{LP}	5.8%	13.9%	

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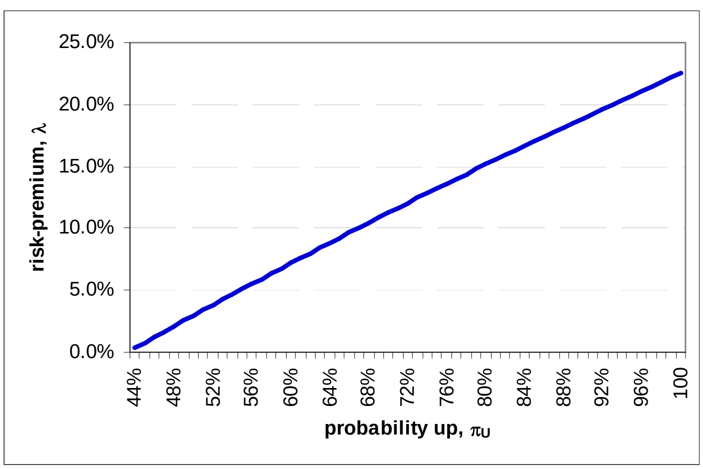
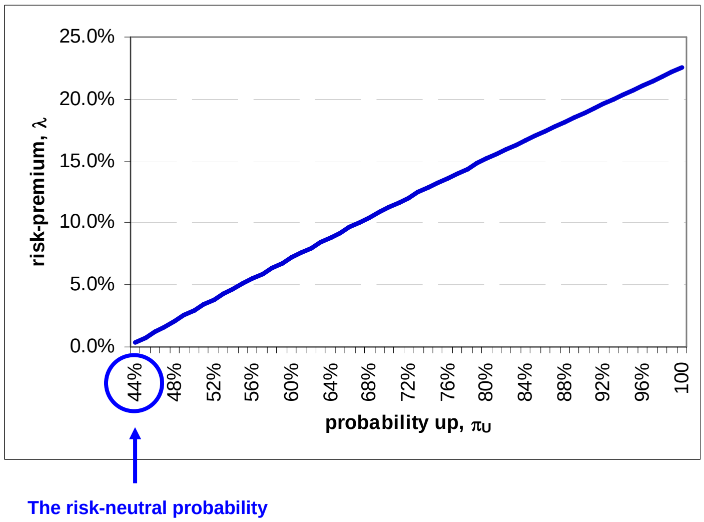
Illustration: Consistent Pairs of Risk-Premium and Probability



All of these pairs of risk premium and probability for the "up" branching yield (i) the same project value, V_P, and (ii) the same product of the probability and the forward state price, $\pi_U \times \varphi_U$.

Indeterminacy (cont.)



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The Risk-Neutral Distribution

- It is the probability distribution which, when combined with an assumption of risk-neutrality, matches the market price.
- It is the product of the true probability distribution and the true forward state prices:

$$\pi_U^* = \pi_U \times \phi_U$$

- The value of any project is the same for a range of true probabilities and true forward state prices. Instead of trying to back out the true forward state prices, we simply back out the product and ignore the decomposition. The product is the risk-neutral distribution.
- In Corporate Risk Management, we often care about true probability distributions and true discount rates, so we will try to decompose the risk-neutral distribution and recover the risk-premium. But we understand the fundamental indeterminacy.

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Forward Prices

Forward Prices

- We did the previous derivation of the risk-neutral probabilities using an arbitrary project with a specified pattern of contingent cash flows. We were not explicit about how those cash flows were calculated from the underlying risk factor.
- For a subset of risk factors, mostly commodity prices as well as financial assets, the benchmark "project" is a forward contract.
- A forward contract has a payoff that is linear in the underlying risk factor.
- A forward price is the certainty-equivalent for the underlying factor:

$$F_t = CEQ_t\left[S_{t+1}\right] = E\left[S_{t+1}\right]e^{-\lambda} = E_t^*\left[S_{t+1}\right] = \left(\pi_U^* S_{t+1,U} + \pi_D^* S_{t+1,D}\right)$$

- The risk-premium derived from a forward price is the risk-premium for any linear payoff.
- The risk neutral probabilities derived from a forward price are the correct discount factors for any payoff contingent on the underlying risk factor.

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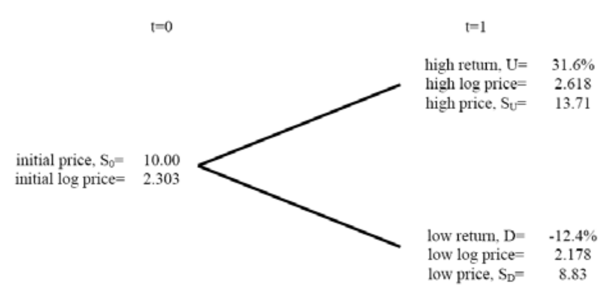
Implementing Risk-Neutral Valuation

Start with the Risk Structure for the Risk Factor S

Figure 6.7
One-Step in a Binomial Tree



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Add the Risk Pricing Information

Add to this risk structure a specified risk-premium, λ=5%. This implies a forward price of \$10.20 and a risk-neutral probability of 39%:

$$F_0 = E[S_1]e^{-\lambda} = (\pi_U\, S_{1,U} + \pi_D\, S_{1,D})e^{-\lambda} = \$10.20.$$

$$\pi_U^* \ solves\ F_0 = E^*[S_1] = (\pi_U^* S_U + \pi_D^* S_D).$$

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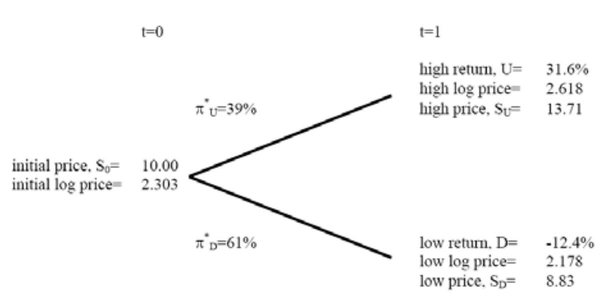
Create an Alternative Tree Using the Risk-Neutral Distribution

Figure 9.4
Revised Binomial Tree: Risk-Neutral Probabilities for the Random Walk



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Example #1: Single-Period, Symmetric Risk

- Suppose you have a one-period project with a payoff proportional to the underlying risk factor, with proportionality factor *q*:
  - $CF_U = q\, S_U$
  - $CF_D = q\, S_D$
- The present value is: $PV_0 = E^*[CF_1]\, e^{r_f} = \left(\pi_U^* CF_U + \pi_D^* CF_D\right) e^{r_f}$

$$= q\,\big(39\%\,(13.04) + 61\%\,(8.40)\big)\,0.961 = q\,(9.80).$$

- Note that this is exactly what you would have gotten using the risk-adjusted discount rate with a risk-premium of 5%:

$$PV_0 = E[CF_1]\, e^{r_a} = \left(\pi_U CF_U + \pi_D CF_D\right) e^{r_a}$$

$$= q\,\big(50\%\,(13.04) + 50\%\,(8.40)\big)\,0.914 = q\,(9.80).$$

- For a project with a linear payoff, there is no advantage to switching to the risk-neutral distribution.

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Example #3: Two-Period, Compounded Risk

- Suppose you have a two-period project with a payoff only at t=2, but still proportional to the underlying risk factor, with proportionality factor *q*:
  - $CF_{UU} = q\, S_{UU}$
  - $CF_{UD} = q\, S_{UD}$
  - $CF_{DD} = q\, S_{DD}$
- The present value is:

$$PV_0 = E^*[CF_2]\, e^{r_f 2} = \left[\pi_U^* \left(\pi_U^* CF_{UU} + \pi_D^* CF_{UD}\right) + \pi_D^* \left(\pi_U^* CF_{UD} + \pi_D^* CF_{DD}\right)\right] e^{r_f 2}$$

$$= q\,\big[39\%\big(39\%\,(17.02) + 61\%\,(10.96)\big) + 61\%\big(39\%\,(10.96) + 61\%\,(7.06)\big)\big]\,0.961^2 = q\,(9.61).$$

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Example #3: Two-Period, Compounded Risk

- Note that this is exactly what you would have gotten using the risk-adjusted discount rate with a risk-premium of 5%, and compounded for two periods:

$$PV_0 = E[CF_2]\,e^{r_a 2} = q\,\left[\pi_U\left(\pi_U CF_{UU} + \pi_D CF_{UD}\right) + \pi_D\left(\pi_U CF_{UD} + \pi_D CF_{DD}\right)\right]e^{r_a 2}$$

$$= q\,\left[50\%\left(50\%\,(17.02) + 50\%\,(10.96)\right) + 50\%\left(50\%\,(17.02) + 50\%\,(10.96)\right)\right]0.914^2$$

$$= q\,(9.60).$$

- Once again, for a project with a linear payoff, <u>and assuming that risk grows linearly through time</u>, there is no advantage to switching to the risk-neutral distribution.

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Risk-Neutral Valuation Exactly Matches Risk-Adjusted Valuation…Sometimes

- When risk is symmetric, and
- When risk grows linearly through time.

- Examples where these conditions do not hold:
  - When is risk not symmetric?
    - Any non-linear payoff structure such as a call option.
  - When does risk not grow linearly through time?
    - When the underlying factor risk does not grow linearly through time – e.g., when it follows a mean reverting process instead of a random walk.
    - When the project's contingent payoff is not a constant function of the underlying factor risk – e.g., when projects are developed or other triggers switch the contingent payoff.

- In these cases
  - The risk-adjusted methodology breaks down or becomes impossibly cumbersome.
  - The risk-neutral methodology is easily applied.

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