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Welcome to a new lecture on the binomial theorem.

In this lecture, we will look at certain divisibility results which can be derived based on the binomial theorem.

Once again, recall the binomial theorem is given by $(x+y)^n = \sum_{r=0}^{n}{C_r x^{n-r} y^r}$, where we use C_r to denote nC_r.

To give you a flavor of the kind of results we will be looking at today, let's look at the first example which asks you to show that $3^{2n+2} - 8n - 9$ is divisible by 64 when n is a natural number.

It asks you to show that this term is divisible by 64 for natural numbers n.

At first sight, there doesn't seem to be a binomial expansion you can apply to generate this result so we would like to create a binomial expansion to derive this result.

Notice that you have an exponent here, but you also have some terms here and these terms are, in general, not divisible by 64 for n being a natural number.

You can plug in certain natural numbers and you can see that this term, which is being subtracted from 3^{2n+2}, is not going to be divisible by 64 and therefore we would like to somehow get rid of these terms and leave the remaining term to be divisible by 64.

Let's look at a way of re-expressing this term.

$3^{2n+2} - 8n - 9$ can be written as 9^{n+1}, where I've just taken a power of 2 into 3 and written 3^2 as 9, and you still have $- 8n - 9$.

So now, this looks a little more promising because you have 9^{n+1} and you have a 9 and you have an 8n.

You can get rid of the 8n potentially by writing 9 as $(1+8)^{n+1}$ and you have $- 8n - 9$.

So this looks a little more promising, and now you have a term which you can expand using the binomial theorem.

Let's try and expand this term.

This is nothing but $(n+1)C_0 1^{n+1} + (n+1)C_1 1^n 8 + (n+1)C_2 1^{n-1} 8^2 + ...$ We'll be concerned about the higher-order terms later, and we have/we're still subtracting 8n and 9 from this term.

Let's see what this simplifies to.

This is nothing but: the first term is 1; the second term is $(n+1)*1*8$; and the remaining terms can be succinctly written as summation \sum_{r=2}^{n+1}{(n+1)C_r 1^{n+1-r} 8^r}.

You still have to subtract 8n and 9 from this expression.

This looks a little more promising, because notice that these first two terms add up to exactly 8n+9 and you can cancel these two terms.

So, you're just left with summation \sum_{r=2}^{n+1}{(n+1)C_r 1^{n+1-r} 8^r}.

This is what we're left with, and we want to show that this is divisible by 64.

Notice that this summation runs from r=2, so let's rewrite this as \sum_{r=0}^{n-1}{(n+1)C_{r+2}8^{r+2}}.

We've just used a change of indices to rewrite this summation.

Notice that there is a 8^2 here; you can pull that out.

This is going to be 64*[\sum_{r=0}^{n-1}{(n+1)C_{r+2} 8^r}].

So you have a 64 here, and you wanted to show that this term is divisible by 64: so that is good.

What we have to show is that this term is an integer, that's all is what remains.

That term is clearly an integer because you have a summation of several terms, where each term is made up of two integers themselves.

8^r is an integer and (n+1)C_{r+2}, again, is an integer.

We have summations of integers and so this term is an integer.

So we have that this overall term is divisible by 64, which is the desired result.

We've seen an application of the binomial theorem where we somehow re-expressed our given term in terms of an expression from where we could use the binomial expansion and derive a divisibility result for the original term.

Let's look at a second example.

This example asks to show that 2^{4n} - (2^n)*(7n+1) is divisible by 196 whenever n is a natural number.

This example is very similar to the previous one, and once again we have to figure out how to rewrite this original term in a form which is amenable to the binomial expansion.

Notice, now, that we want to show that this term is divisible by 196, and 196 is nothing but $(14)^2$ and you have a 7 here and a 2^n here, so that looks promising.

Let's see what we can do.

$2^{4n} - (2^n)*(7n+1)$ can be rewritten as $(16)^n - (2^n)*(7n+1)$, where we've just rewritten 2^{4n} as $(16)^n$.

Now, 16 can somehow be split into 14 and 2; so, we can rewrite this as $(2+14)^n - (2^n)*(7n+1)$.

This looks promising because, now, we have a 14 appear in the binomial term and we want to show that this whole term is divisible by 196 which is $(14)^2$.

Let's go through the motions again and use the binomial theorem to show that this is nothing but $nC_0 2^n (14)^0 + nC_1 2^{n-1} (14)^1$ + (let's rewrite the higher order terms as) summation $\sum_{r=2}^{n}{nC_r 2^{n-r} (14)^r}$,and we subtract the original $(2^n)*(7n+1)$ from this expansion.

Once again, you can rewrite this as $2^n + n*(2^{n-1})*14 + \sum_{r=2}^{n}{nC_r 2^{n-r} (14)^r}$ and you still have to subtract $(2^n)*(7n+1)$.

This is nothing but $2^n + n*(2^n)*7$, where I've factorized 14 as $2*7$, and you have a resulting summation $\sum{r=2}^{n}{nC_r 2^{n-r} (14)^r}$, and you subtract $(2^n)*7n$ and a 2^n, where I've just expanded this term.

Notice once again that you can cancel the first two terms with the terms being subtracted, and you're still left with/you're only left with the summation $\sum_{r=2}^{n}{nC_r 2^{n-r} (14)^r}$.

Now the arguments are quite similar to the previous one/previous example we've seen, so I won't go through this in detail.

You can just rewrite this as $(14^2)*[\sum_{r=0}^{n-2}{nC_{r+2} 2^{n-r-2} (14)^r}]$, where here you started with 2^{n-2} so you're starting with 2^{n-2} again; and here you started with $(14)^2$ whereas here you start with $(14)^0$ since I've pulled out a $(14)^2$ here.

This is what we're left with, and notice that this summation again is an integer and we have this summation being multiplied by $(14)^2$ which is 196 and which is again an integer.

We wanted to show that this term is divisible by 196, and that's exactly what we've shown since we've factorized it into a form where we have a leading coefficient of $(14)^2$ and so, we've once again shown that this term is divisible by 196 whenever n is a natural number.

Let's look at a multiple choice question which is based on this flavor of problems.

The question asks you to show/asks you when 5^{99} is divided by 13, the remainder is: it asks you to compute the remainder when 5^99 is divided by 13.

The options are 8, 9, 10, and "none of these".

This is a kind of problem where it's hard to eliminate the options without actually solving the problem because you have "none of these".

Even if you determine that this remainder is going to be even or odd, you are not really sure which option it is since there's a "none of these" and, even if the remainder is odd, you can't just conclude that the answer is 9.

So you would actually have to derive the result for this example.

Let's see how we can rewrite 5^{99} to obtain an amenable result.

To arrive at the solution, let's try computing some powers of 5 to see where that gets us.

5^1 is 5, which is quite far from 13: there's not really much you can do about it.

5^2 is 25, which is close to a multiple of 13 (13*2 is 26, so this looks promising).

But 5^{99} is not a direct power of/not an integral power of 5^2, and so let's try this approach: let's write 5^{99} as 5*(5^{98}) where I've just factorized 5^{99} into a more suitable form.

Now I can write this as 5*((5^2)^{49}).

Since I know that 5^2 is close to a multiple of 13, this is nothing but 5*(26 - 1)^{49}.

26 is a multiple of 13, so the divisibility results are going to be fairly easy to derive.

This is nothing but 5*..., and now you use the binomial expansion of (26-1)^{49}.

Let's see how we can write this.

5*[49C_0 (-1)^{49} (26)^0 (I'm starting off with (-1)^{49} and (26)^0 since we're only concerned with the power of 26 which is 0, since all other terms are going to be divisible by 13) and you have + 49C_1 (-1)^{48} (26)^1 + ...]. This is going to be the expansion of (26-1)^{49}, and you have a leading term which is 5 in this expression.

Notice that all of these higher powers, as I mentioned, are going to be divisible by 13, and, since we only care about the remainder that this term leaves when divided by 13, we don't really care about these terms which are exactly divisible by 13.

This is the only remaining term, which is going to give us the remainder when you divide 5^{99} by 13 so let's see what that gives us.

$5*$[$49C_0$, which is 1, times $(-1)^{49}$, which is -1, times $(26)^0$, which is 1], and so this gives us -5.

Therefore, the remainder when 5^{99} is divided by 13 is -5.

For those of you who aren't used to this kind of notation, this might seem a little strange that you get a negative remainder but, to get a positive remainder, you just add a multiple of 13.

Since we've ignored so many multiples of 13, you get a negative number but, if you add a multiple of 13, you get 8.

The remainder when 5^{99} is divided by 13 is 8.

For those of you who are familiar with modular arithmetic, notice that the approach we took is very similar to the approach you take when you use/when you do such kinds of problems in number theory where you try to get at a power of 5 which is close to a multiple of 13 and you just use remainder arguments, where you ignore terms which were multiples of 13, and so you're only left with one term since you only care about the remainder.

This is a sample of a problem which could appear in an exam-type situation where you are asked to derive a result based on the divisibility of a term by a natural number using the binomial theorem.

That's it for this lecture.

Hope you enjoyed watching this lecture.

In the next lectures, we will look at some example problems based on the binomial theorem.

I thank you for your attention, and hope you can join us next time.