28 $\int_{1}^{4} \dfrac{dx}{\sqrt{x^2 - 1}}$

29 $\int_{2}^{x} \dfrac{dx}{(x^2 - 1)^{3/2}}$

30 $\int_{-1}^{1} \dfrac{x\, dx}{x^2 + 1}$

31 $\int_{-x}^{x} \dfrac{dx}{x^2 + 9}$

**32** $\int_{1.2}^{1} \sqrt{1 - x^2}\, dx$ = area of _____.

33 Combine the integrals to prove the reduction formula $(n \neq 0)$:

$$\int \frac{x^{n-1}}{x^2 + 1}\, dx = \frac{x^n}{n} - \int \frac{x^{n-1}}{x^2 + 1}\, dx.$$

34 Integrate $1/\cos x$ and $1/(1 + \cos x)$ and $\sqrt{1 + \cos x}$.

35 (a) $x =$ _____ gives $\int dx/\sqrt{x^2 - 1} = \ln(\sec \theta + \tan \theta)$.
(b) From the triangle, this answer is $f = \ln(x + \sqrt{x^2 - 1})$. Check that $df/dx = 1/\sqrt{x^2 - 1}$.
(c) Verify that $\cosh f = \frac{1}{2}(e^f + e^{-f}) = x$. Then $f = \cosh^{-1} x$, the answer in Example 8.
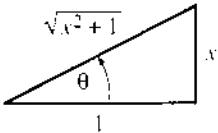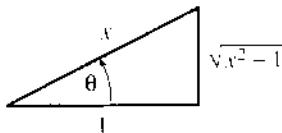
36 (a) $x =$ _____ gives $\int dx/\sqrt{x^2 + 1} = \ln(\sec \theta + \tan \theta)$.
(b) The second triangle converts this answer to $g = \ln(x + \sqrt{x^2 + 1})$. Check that $dg/dx = 1/\sqrt{x^2 + 1}$.
(c) Verify that $\sinh g = \frac{1}{2}(e^g - e^{-g}) = x$ so $g = \sinh^{-1} x$.
(d) Substitute $x = \sinh g$ directly into $\int dx/\sqrt{x^2 + 1}$ and integrate.

In **37–42** substitute $x = \sinh \theta$, $\cosh \theta$, or $\tanh \theta$. After integration change back to x.

37 $\int \dfrac{dx}{\sqrt{x^2 - 1}}$

38 $\int \dfrac{dx}{x\sqrt{1 - x^2}}$

39 $\int \sqrt{x^2 - 1}\, dx$

40 $\int \dfrac{\sqrt{x^2 - 1}}{x^2}\, dx$

41 $\int \dfrac{dx}{1 - x^2}$

42 $\int \dfrac{\sqrt{1 + x^2}}{x^2}\, dx$

Rewrite **43–48** as $(x + b)^2 + C$ or $-(x - b)^2 + C$ by completing the square.

43 $x^2 - 4x + 8$

44 $-x^2 + 2x + 8$

45 $x^2 - 6x$

46 $-x^2 + 10$

47 $x^2 + 2x + 1$

48 $x^2 + 4x - 12$

49 For the three functions $f(x)$ in Problems 43, 45, 47 integrate $1/f(x)$.

50 For the three functions $g(x)$ in Problems 44, 46, 48 integrate $1/\sqrt{g(x)}$.

51 For $\int dx/(x^2 + 2bx + c)$ why does the answer have different forms for $b^2 > c$ and $b^2 < c$? What is the answer if $b^2 = c$?

52 What substitution $u = x + b$ or $u = x - b$ will remove the linear term?

(a) $\int \dfrac{dx}{x^2 - 4x + c}$

(b) $\int \dfrac{dx}{3x^2 + 6x}$

(c) $\int \dfrac{dx}{-x^2 + 10x + c}$

(d) $\int \dfrac{dx}{2x^2 - x}$

53 *Find the mistake.* With $x = \sin \theta$ and $\sqrt{1 - x^2} = \cos \theta$, substituting $dx = \cos \theta\, d\theta$ changes

$$\int_{0}^{2\pi} \cos^2 \theta\, d\theta \quad \text{into} \quad \int_{0}^{0} \sqrt{1 - x^2}\, dx.$$

54 (a) If $x = \tan \theta$ then $\int \sqrt{1 + x^2}\, dx = \int$ _____ $d\theta$.
(b) Convert $\frac{1}{2}[\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta)]$ back to x.
(c) If $x = \sinh \theta$ then $\int \sqrt{1 + x^2}\, dx = \int$ _____ $d\theta$.
(d) Convert $\frac{1}{2}[\sinh \theta \cosh \theta + \theta]$ back to x.

These answers agree. In Section 8.2 they will give the length of a parabola. Compare with Problem 7.2.62.

55 Rescale x and y in Figure 7.5b to produce the equal area $\int y\, dx$ in Figure 7.5c. What happens to y and what happens to dx?

56 Draw $y = 1 \cdot \sqrt{1 - x^2}$ and $y = 1 \cdot \sqrt{16 - x^2}$ to the same scale (1″ across and up; 4″ across and $\frac{1}{4}$″ up).

57 What is wrong, if anything, with

$$\int_{0}^{2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} 2?$$

7.4 Partial Fractions

This section is about rational functions $P(x)/Q(x)$. Sometimes their integrals are also rational functions (ratios of polynomials). More often they are not. It is very common for the integral of P/Q to involve logarithms. We meet logarithms immediately in the

simple case $1/(x-2)$, whose integral is $\ln|x-2| + C$. We meet them again in a sum of simple cases:

$$\int \left[\frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x} \right] dx = \ln|x-2| + 3\ln|x+2| - 4\ln|x| + C.$$

Our plan is to split P/Q into a sum like this—and integrate each piece.
Which rational function produced that particular sum? It was

$$\frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x} = \frac{(x+2)(x) + 3(x-2)(x) - 4(x-2)(x+2)}{(x-2)(x+2)(x)} = \frac{-4x+16}{(x-2)(x+2)(x)}.$$

This is P/Q. It is a ratio of polynomials, degree 1 over degree 3. The pieces of P are collected into $-4x + 16$. The common denominator $(x-2)(x+2)(x) = x^3 - 4x$ is Q. But I kept these factors separate, for the following reason. When we start with P/Q, and break it into a sum of pieces, *the first things we need are the factors of Q.*

In the standard problem P/Q is given. To integrate it, we break it up. The goal of partial fractions is to find the pieces—*to prepare for integration.* That is the technique to learn in this section, and we start right away with examples.

EXAMPLE 1 Suppose P/Q has the same Q but a different numerator P:

$$\frac{P}{Q} = \frac{3x^2 + 8x - 4}{(x-2)(x+2)(x)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}. \tag{1}$$

Notice the form of those pieces! They are the *"partial fractions"* that add to P/Q. Each one is a constant divided by a factor of Q. We know the factors $x-2$ and $x+2$ and x. We don't know the constants A, B, C. In the previous case they were $1, 3, -4$. In this and other examples, there are two ways to find them.

Method 1 (slow) Put the right side of (1) over the common denominator Q:

$$\frac{3x^2 + 8x - 4}{Q} = \frac{A(x+2)(x) + B(x-2)(x) + C(x-2)(x+2)}{(x-2)(x+2)(x)} \tag{2}$$

Why is A multiplied by $(x+2)(x)$? Because canceling those factors will leave $A/(x-2)$ as in equation (1). Similarly we have $B/(x+2)$ and C/x. *Choose the numbers A, B, C so that the numerators match.* As soon as they agree, the splitting is correct.

Method 2 (quicker) Multiply equation (1) by $x-2$. That leaves a space:

$$\frac{3x^2 + 8x - 4}{(x+2)(x)} = A + \frac{B(x-2)}{x+2} + \frac{C(x-2)}{x}. \tag{3}$$

Now set $x = 2$ and immediately you have A. The last two terms of (3) are zero, because $x-2$ is zero when $x=2$. On the left side, $x=2$ gives

$$\frac{3(2)^2 + 8(2) - 4}{(2+2)(2)} = \frac{24}{8} = 3 \quad \text{(which is } A\text{)}.$$

Notice how multiplying by $x-2$ produced a hole on the left side. Method 2 is the *"cover-up method."* *Cover up $x-2$ and then substitute* $x=2$. The result is $3 = A + 0 + 0$, just what we wanted.

In Method 1, the numerators of equation (2) must agree. The factors that multiply B and C are again zero at $x=2$. That leads to the same A—but the cover-up method avoids the unnecessary step of writing down equation (2).

Calculation of B   Multiply equation (1) by $x + 2$, which covers up the $(x + 2)$:

$$\frac{3x^2 + 8x - 4}{(x - 2)\quad(x)} = \frac{A(x + 2)}{x - 2} + B + \frac{C(x + 2)}{x}. \tag{4}$$

Now set $x = -2$, so A and C are multiplied by zero:

$$\frac{3(-2)^2 + 8(-2) - 4}{(-2 - 2)\quad(-2)} = \frac{-8}{8} = -1 = B.$$

This is almost full speed, but (4) was not needed. *Just cover up in Q and give x the right value* (which makes the covered factor zero).

Calculation of C (quickest)   In equation (1), cover up the factor (x) and set $x = 0$:

$$\frac{3(0)^2 + 8(0) - 4}{(0 - 2)(0 + 2)} = \frac{-4}{-4} = 1 = C. \tag{5}$$

To repeat: The same result $A = 3$, $B = -1$, $C = 1$ comes from Method 1.

EXAMPLE 2 $$\frac{x + 2}{(x - 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 3}.$$

First cover up $(x - 1)$ on the left and set $x = 1$. Next cover up $(x + 3)$ and set $x = -3$:

$$\frac{1 + 2}{(\quad)(1 + 3)} = \frac{3}{4} = A \qquad \frac{-3 + 2}{(-3 - 1)(\quad)} = \frac{-1}{-4} = B.$$

The integral is $\frac{3}{4}\ln|x - 1| + \frac{1}{4}\ln|x + 3| + C$.

EXAMPLE 3   This was needed for the logistic equation in Section 6.5:

$$\frac{1}{y(c - by)} = \frac{A}{y} + \frac{B}{c - by}. \tag{6}$$

First multiply by y. That covers up y in the first two terms and changes B to By. Then set $y = 0$. The equation becomes $1/c = A$.

To find B, multiply by $c - by$. That covers up $c - by$ in the outside terms. In the middle, A times $c - by$ will be zero at $y = c/b$. That leaves B on the right equal to $1/y = b/c$ on the left. Then $A = 1/c$ and $B = b/c$ give the integral announced in Equation 6.5.9:

$$\int \frac{dy}{cy - by^2} = \int \frac{dy}{cy} + \int \frac{b\,dy}{c(c - by)} = \frac{\ln y}{c} - \frac{\ln(c - by)}{c}. \tag{7}$$

It is time to admit that the general method of partial fractions can be very awkward. First of all, it requires the factors of the denominator Q. When Q is a quadratic $ax^2 + bx + c$, we can find its roots and its factors. In theory a cubic or a quartic can also be factored, but in practice only a few are possible—for example $x^4 - 1$ is $(x^2 - 1)(x^2 + 1)$. Even for this good example, *two of the roots are imaginary.* We can split $x^2 - 1$ into $(x + 1)(x - 1)$. We cannot split $x^2 + 1$ without introducing i.

The method of partial fractions can work directly with $x^2 + 1$, as we now see.

EXAMPLE 4 $\int \frac{3x^2 + 2x + 7}{x^2 + 1}\,dx$ (a quadratic over a quadratic).

This has another difficulty. The degree of P equals the degree of Q (= 2). *Partial*

fractions cannot start until P has lower degree. Therefore I divide the leading term x^2 into the leading term $3x^2$. That gives 3, which is separated off by itself:

$$\frac{3x^2 + 2x + 7}{x^2 + 1} = 3 + \frac{2x + 4}{x^2 + 1}. \tag{8}$$

Note how 3 really used $3x^2 + 3$ from the original numerator. That left $2x + 4$. *Partial fractions will accept a linear factor* $2x + 4$ (or $Ax + B$, not just A) *above a quadratic*.

This example contains $2x/(x^2 + 1)$, which integrates to $\ln(x^2 + 1)$. The final $4/(x^2 + 1)$ integrates to $4 \tan^{-1}x$. When the denominator is $x^2 + x + 1$ we complete the square before integrating. The point of Sections 7.2 and 7.3 was to make that integration possible. This section gets the fraction ready—in parts.

The essential point is that we never have to go higher than quadratics. *Every denominator Q can be split into linear factors and quadratic factors*. There is no magic way to find those factors, and most examples begin by giving them. They go into their own fractions, and they have their own numerators—which are the A and B and $2x + 4$ we have been computing.

The one remaining question is *what to do if a factor is repeated*. This happens in Example 5.

EXAMPLE 5
$$\frac{2x + 3}{(x - 1)^2} = \frac{A}{(x - 1)} + \frac{B}{(x - 1)^2}.$$

The key is the new term $B/(x - 1)^2$. That is the right form to expect. With $(x - 1)(x - 2)$ this term would have been $B/(x - 2)$. But when $(x - 1)$ is repeated, something new is needed. To find B, *multiply through by* $(x - 1)^2$ *and set* $x = 1$:

$$2x + 3 = A(x - 1) + B \quad \text{becomes} \quad 5 = B \quad \text{when} \quad x = 1.$$

This cover-up method gives B. Then $A = 2$ is easy, and the integral is $2 \ln|x - 1| - 5/(x - 1)$. The fraction $5/(x - 1)^2$ has an integral without logarithms.

EXAMPLE 6
$$\frac{2x^3 + 9x^2 + 4}{x^2(x^2 + 4)(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 4} + \frac{E}{x - 1}.$$

This final example has almost everything! It is more of a game than a calculus problem. In fact calculus doesn't enter until we integrate (and nothing is new there). Before computing A, B, C, D, E, we write down the overall rules for partial fractions:

1. The degree of P must be less than the degree of Q. Otherwise divide their leading terms as in equation (8) to lower the degree of P. Here $3 < 5$.
2. Expect the fractions illustrated by Example 6. The linear factors x and $x + 1$ (*and the repeated* x^2) are underneath constants. The quadratic $x^2 + 4$ is under a linear term. A repeated $(x^2 + 4)^2$ would be under a new $Fx + G$.
3. Find the numbers A, B, C, ... by any means, including cover-up.
4. Integrate each term separately and add.

We could prove that this method always works. It makes better sense to show that it works once, in Example 6.

To find E, cover up $(x - 1)$ on the left and substitute $x = 1$. Then $E = 3$. To find B, cover up x^2 on the left and set $x = 0$. Then $B = 4/(0 + 4)(0 - 1) = -1$. The cover-up method has done its job, and there are several ways to find A, C, D.

Compare the numerators, after multiplying through by the common denominator Q:

$$2x^3 + 9x^2 + 4 = Ax(x^2 + 4)(x - 1) - (x^2 + 4)(x - 1) + (Cx + D)(x^2)(x - 1) + 3x^2(x^2 + 4).$$

The known terms on the right, from $B = -1$ and $E = 3$, can move to the left:

$$-3x^4 + 3x^3 - 4x^2 + 4x = Ax(x^2 + 4)(x - 1) + (Cx + D)x^2(x - 1).$$

We can divide through by x and $x - 1$, which checks that B and E were correct:

$$-3x^2 - 4 = A(x^2 + 4) + (Cx + D)x.$$

Finally $x = 0$ yields $A = -1$. This leaves $-2x^2 = (Cx + D)x$. Then $C = -2$ and $D = 0$.

You should never have to do such a problem! I never intend to do another one. It completely depends on expecting the right form and matching the numerators. They could also be matched by comparing coefficients of $x^4, x^3, x^2, x, 1$—to give five equations for A, B, C, D, E. That is an invitation to human error. Cover-up is the way to start, and usually the way to finish. With repeated factors and quadratic factors, match numerators at the end.

7.4 EXERCISES

Read-through questions

The idea of __a__ fractions is to express $P(x)/Q(x)$ as a __b__ of simpler terms, each one easy to integrate. To begin, the degree of P should be __c__ the degree of Q. Then Q is split into __d__ factors like $x - 5$ (possibly repeated) and quadratic factors like $x^2 + x + 1$ (possibly repeated). The quadratic factors have two __e__ roots, and do not allow real linear factors.

A factor like $x - 5$ contributes a fraction $A/$__f__. Its integral is __g__. To compute A, cover up __h__ in the denominator of P/Q. Then set $x =$ __i__, and the rest of P/Q becomes A. An equivalent method puts all fractions over a common denominator (which is __j__). Then match the __k__. At the same point $x =$ __l__ this matching gives A.

A repeated linear factor $(x - 5)^2$ contributes not only $A/(x - 5)$ but also $B/$__m__. A quadratic factor like $x^2 + x + 1$ contributes a fraction __n__$/(x^2 + x + 1)$ involving C and D. A repeated quadratic factor or a triple linear factor would bring in $(Ex + F)/(x^2 + x + 1)^2$ or $G/(x - 5)^3$. The conclusion is that any P/Q can be split into partial __o__, which can always be integrated.

1 Find the numbers A and B to split $1/(x^2 - x)$:

$$\frac{1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}.$$

Cover up x and set $x = 0$ to find A. Cover up $x - 1$ and set $x = 1$ to find B. Then integrate.

2 Find the numbers A and B to split $1/(x^2 - 1)$:

$$\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}.$$

Multiply by $x - 1$ and set $x = 1$. Multiply by $x + 1$ and set $x = -1$. *Integrate.* Then find A and B again by method 1—with numerator $A(x + 1) + B(x - 1)$ equal to 1.

Express the rational functions 3–16 as partial fractions:

3 $\dfrac{1}{(x - 3)(x - 2)}$

4 $\dfrac{x}{(x - 3)(x - 2)}$

5 $\dfrac{x^2 + 1}{(x)(x + 1)(x + 2)}$

6 $\dfrac{1}{x^3 - x}$

7 $\dfrac{3x + 1}{x^2}$

8 $\dfrac{3x + 1}{(x - 1)^2}$

9 $\dfrac{3x^2}{x^2 + 1}$ (divide first)

10 $\dfrac{1}{(x - 1)(x^2 + 1)}$

11 $\dfrac{1}{x^2(x - 1)}$

12 $\dfrac{x}{x^2 - 4}$

13 $\dfrac{1}{x(x - 1)(x - 2)(x - 3)}$

14 $\dfrac{x^2 + 1}{x + 1}$ (divide first)

15 $\dfrac{1}{x^4 - 1} = \dfrac{1}{(x + 1)(x - 1)(x^2 + 1)}$

16 $\dfrac{1}{x^2(x - 1)}$ $\left(\text{remember the } \dfrac{A}{x} \text{ term}\right)$

17 Apply Method 1 (matching numerators) to Example 3:

$$\frac{1}{cy - by^2} = \frac{A}{y} + \frac{B}{c - by} = \frac{A(c - by) + By}{y(c - by)}.$$

Match the numerators on the far left and far right. Why does $Ac = 1$? Why does $-bA + B = 0$? What are A and B?

18 What goes wrong if we look for A and B so that

$$\frac{x^2}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}?$$

Over a common denominator, try to match the numerators. What to do first?

19 Split $\dfrac{3x^2}{x^3-1} = \dfrac{3x^2}{(x-1)(x^2+x+1)}$ into $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$.

(a) Cover up $x-1$ and set $x=1$ to find A.

(b) Subtract $A/(x-1)$ from the left side. Find $Bx+C$.

(c) Integrate all terms. Why do we already know

$$\ln(x^3-1) = \ln(x-1) + \ln(x^2+x+1)?$$

20 Solve $dy/dt = 1-y^2$ by separating $\int dy/1-y^2 = \int dt$. Then

$$\frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)} = \frac{1/2}{1-y} + \frac{1/2}{1+y}$$

Integration gives $\frac{1}{2}\ln\rule{1.5cm}{0.4pt} = t+C$. With $y_0 = 0$ the constant is $C = \rule{1cm}{0.4pt}$. Taking exponentials gives $\rule{1.5cm}{0.4pt}$. The solution is $y = \rule{1cm}{0.4pt}$. *This is the S-curve.*

By substitution change 21–28 to integrals of rational functions. Problem 23 integrates $1/\sin\theta$ with no special trick.

21 $\displaystyle\int \frac{e^x dx}{e^{2x} - e^x}$

22 $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

23 $\displaystyle\int \frac{\sin\theta\,d\theta}{1-\cos^2\theta}$

24 $\displaystyle\int \frac{dt}{(e^t - e^{-t})^2}$

25 $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$

26 $\displaystyle\int \frac{\sqrt[3]{x}-8}{x}\,dx$

27 $\displaystyle\int \frac{dx}{1+\sqrt{x+1}}$

28 $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

29 Multiply this partial fraction by $x-a$. Then let $x \to a$:

$$\frac{1}{Q(x)} = \frac{A}{x-a} + \cdots.$$

Show that $A = 1/Q'(a)$. When $x = a$ is a double root this fails because $Q'(a) = \rule{1.5cm}{0.4pt}$.

30 Find A in $\dfrac{1}{x^8-1} = \dfrac{A}{x-1} + \cdots$. Use Problem 29.

31 (for instructors only) Which rational functions P/Q are the derivatives of other rational functions (no logarithms)?

7.5 Improper Integrals

"Improper" means that some part of $\int_a^b y(x)dx$ becomes *infinite*. It might be b or a or the function y. The region under the graph reaches infinitely far—to the right or left or up or down. (Those come from $b = \infty$ and $a = -\infty$ and $y \to \infty$ and $y \to -\infty$.) Nevertheless the integral may *"converge."* Just because the *region* is infinite, it is not automatic that the *area* is infinite. That is the point of this section—to decide when improper integrals have proper answers.

The first examples show finite area when $b = \infty$, then $a = -\infty$, then $y = 1/\sqrt{x}$ at $x = 0$. The areas in Figure 7.6 are 1, 1, 2:

$$\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\bigg]_1^\infty = 1 \qquad \int_{-\infty}^0 e^x dx = e^x\bigg]_{-\infty}^0 = 1 \qquad \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\bigg]_0^1 = 2.$$
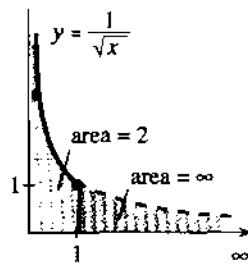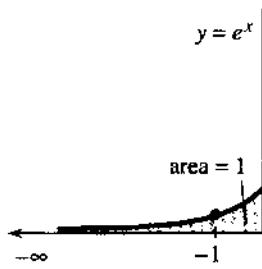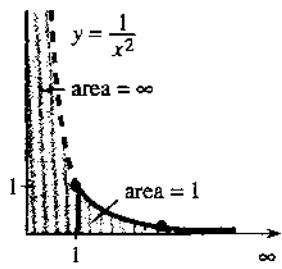

Fig. 7.6 The shaded areas are finite but the regions go to infinity.

In practice we substitute the dangerous limits and watch what happens. When the integral is $-1/x$, substituting $b = \infty$ gives "$-1/\infty = 0$." When the integral is e^x, substituting $a = -\infty$ gives "$e^{-\infty} = 0$." I think that is fair, and I know it is successful. But it is not completely precise.

The strict rules involve a limit. Calculus sneaks up on $1/\infty$ and $e^{-\infty}$ just as it sneaks up on $0/0$. Instead of swallowing an infinite region all at once, the formal definitions push out to the limit:

DEFINITION $\displaystyle\int_a^\infty y(x)dx = \lim_{b\to\infty}\int_a^b y(x)dx \qquad \int_{-\infty}^b y(x)dx = \lim_{a\to-\infty}\int_a^b y(x)dx.$

The conclusion is the same. The first examples converged to 1, 1, 2. Now come two more examples going out to $b = \infty$:

The area under $1/x$ is *infinite*: $\displaystyle\int_1^\infty \frac{dx}{x} = \ln x\Big]_1^\infty = \infty \tag{1}$

The area under $1/x^p$ is *finite* if $p > 1$: $\displaystyle\int_1^\infty \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}\Big]_1^\infty = \frac{1}{p-1}. \tag{2}$

The area under $1/x$ is like $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which is also infinite. In fact the sum approximates the integral—the curved area is close to the rectangular area. They go together (slowly to infinity).

A larger p brings the graph more quickly to zero. Figure 7.7a shows a finite area $1/(p-1) = 100$. The region is still infinite, but we can cover it with strips cut out of a square! The borderline for finite area is $p = 1$. I call it the borderline, but $p = 1$ is strictly on the side of divergence.

The borderline is also $p = 1$ when the function climbs the y axis. At $x = 0$, the graph of $y = 1/x^p$ goes to infinity. For $p = 1$, the area under $1/x$ is again infinite. But at $x = 0$ it is a *small p* (meaning $p < 1$) that produces finite area:

$$\int_0^1 \frac{dx}{x} = \ln x\Big]_0^1 = \infty \qquad \int_0^1 \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}\Big]_0^1 = \frac{1}{1-p} \quad \text{if } p < 1. \tag{3}$$

Loosely speaking "$-\ln 0 = \infty$." Strictly speaking we integrate from the point $x = a$ *near* zero, to get $\int_a^1 dx/x = -\ln a$. As a approaches zero, the area shows itself as infinite. For $y = 1/x^2$, which blows up faster, the area $-1/x]_0^1$ is again infinite.

For $y = 1/\sqrt{x}$, the area from 0 to 1 is 2. In that case $p = \frac{1}{2}$. For $p = 99/100$ the area is $1/(1-p) = 100$. Approaching $p = 1$ the borderline in Figure 7.7 seems clear. *But that cutoff is not as sharp as it looks.*
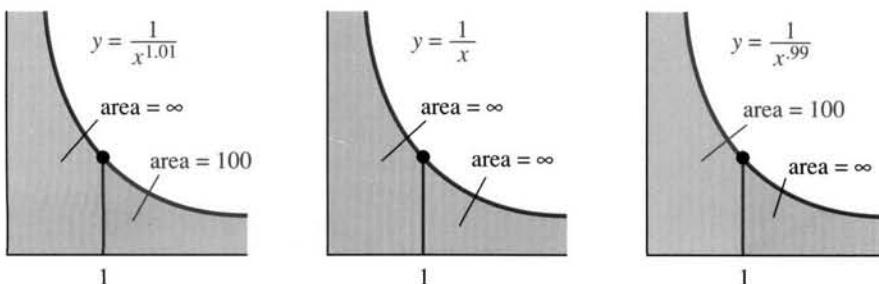

Fig. 7.7 Graphs of $1/x^p$ on both sides of $p = 1$. I drew the same curves!

Narrower borderline Under the graph of $1/x$, the area is infinite. When we divide by $\ln x$ or $(\ln x)^2$, the borderline is somewhere in between. One has infinite area (going out to $x = \infty$), the other area is finite:

$$\int_e^\infty \frac{dx}{x(\ln x)} = \ln(\ln x) \bigg]_e^\infty = \infty \qquad \int_e^\infty \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x} \bigg]_e^\infty = 1. \tag{4}$$

The first is $\int du/u$ with $u = \ln x$. The logarithm of $\ln x$ does eventually make it to infinity. At $x = 10^{10}$, the logarithm is near 23 and $\ln(\ln x)$ is near 3. That is slow! Even slower is $\ln(\ln(\ln x))$ in Problem 11. No function is *exactly* on the borderline.

The second integral in equation (4) is convergent (to 1). It is $\int du/u^2$ with $u = \ln x$. At first I wrote it with x going from zero to infinity. That gave an answer I couldn't believe:

$$\int_0^\infty \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x} \bigg]_0^\infty = 0 \text{ (??)}$$

There must be a mistake, because we are integrating a positive function. The area can't be zero. It is true that $1/\ln b$ goes to zero as $b \to \infty$. It is also true that $1/\ln a$ goes to zero as $a \to 0$. *But there is another infinity in this integral.* The trouble is at $x = 1$, where $\ln x$ is zero and the area is infinite.

EXAMPLE 1 The factor e^{-x} overrides any power x^p (but only as $x \to \infty$).

$$\int_0^\infty x^{50} e^{-x} dx = 50! \quad \text{but} \quad \int_0^\infty x^{-1} e^{-x} dx = \infty.$$

The first integral is $(50)(49)(48) \cdots (1)$. It comes from fifty integrations by parts (not recommended). Changing 50 to $\frac{1}{2}$, the integral defines "$\frac{1}{2}$ *factorial*." The product $\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2}) \cdots$ has no way to stop, but somehow $\frac{1}{2}!$ is $\frac{1}{2}\sqrt{\pi}$. See Problem 28. The integral $\int_0^\infty x^0 e^{-x} dx = 1$ is the reason behind "*zero factorial*" $= 1$. That seems the most surprising of all.

The area under e^{-x}/x is $(-1)! = \infty$. The factor e^{-x} is absolutely no help at $x = 0$. That is an example (the first of many) in which we do not know an antiderivative— but still we get a decision. To integrate e^{-x}/x we need a computer. But to decide that an improper integral is infinite (in this case) or finite (in other cases), we rely on the following *comparison test*:

7C (*Comparison test*) Suppose that $0 \leqslant u(x) \leqslant v(x)$. Then the area under $u(x)$ is smaller than the area under $v(x)$:

$$\int u(x) dx < \infty \text{ if } \int v(x) dx < \infty \qquad \text{if } \int u(x) dx = \infty \text{ then } \int v(x) dx = \infty.$$

Comparison can decide if the area is finite. We don't get the exact area, but we learn about one function from the other. The trick is to construct a simple function (like $1/x^p$) which is on one side of the given function—and stays close to it:

EXAMPLE 2 $\qquad \displaystyle\int_1^\infty \frac{dx}{x^2 + 4x}$ converges by comparison with $\displaystyle\int_1^\infty \frac{dx}{x^2} = 1.$

EXAMPLE 3 $\qquad \displaystyle\int_1^\infty \frac{dx}{\sqrt{x} + 1}$ diverges by comparison with $\displaystyle\int_1^\infty \frac{dx}{2\sqrt{x}} = \infty.$

EXAMPLE 4 $\displaystyle\int_0^1 \frac{dx}{x^2 + 4x}$ diverges by comparison with $\displaystyle\int_0^1 \frac{dx}{5x} = \infty$.

EXAMPLE 5 $\displaystyle\int_0^1 \frac{dx}{\sqrt{x} + 1}$ converges by comparison with $\displaystyle\int_0^1 \frac{dx}{1} = 1$.

In Examples 2 and 5, the integral on the right is larger than the integral on the left. Removing $4x$ and $\sqrt{x}$ increased the area. Therefore the integrals on the left are somewhere between 0 and 1.

In Examples 3 and 4, we *increased* the denominators. The integrals on the right are smaller, but still they diverge. So the integrals on the left diverge. The idea of *comparing functions* is seen in the next examples and Figure 7.8.

EXAMPLE 6 $\displaystyle\int_0^\infty e^{-x^2} dx$ is below $\displaystyle\int_0^1 1\, dx + \int_1^\infty e^{-x} dx = 1 + 1$.

EXAMPLE 7 $\displaystyle\int_1^e \frac{dx}{\ln x}$ is above $\displaystyle\int_1^e \frac{dx}{x \ln x} = \infty$.

EXAMPLE 8 $\displaystyle\int_0^1 \frac{dx}{\sqrt{x - x^2}}$ is below $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} + \int_0^1 \frac{dx}{\sqrt{1 - x}} = 2 + 2$.
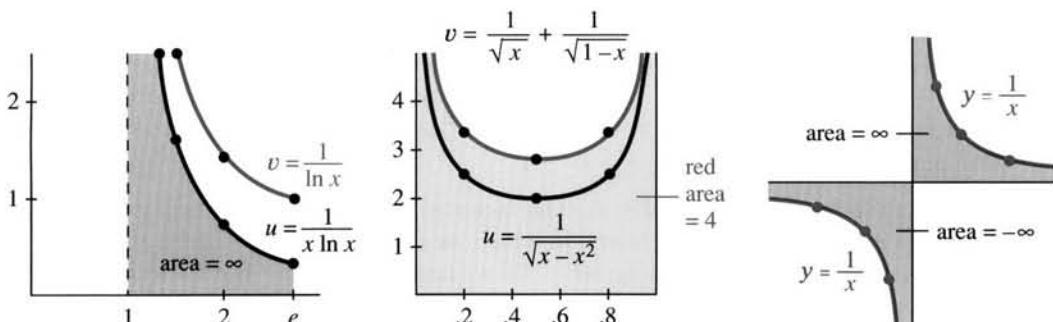

Fig. 7.8 Comparing $u(x)$ to $v(x)$: $\int_1^e dx/\ln x = \infty$ and $\int_0^1 dx/\sqrt{x - x^2} < 4$. But $\infty - \infty \neq 0$.

There are two situations not yet mentioned, and both are quite common. The first is an integral all the way from $a = -\infty$ to $b = +\infty$. That is split into two parts, and **each part must converge**. By definition, the limits at $-\infty$ and $+\infty$ are kept *separate*:

$$\int_{-\infty}^\infty y(x)\, dx = \int_{-\infty}^0 y(x)\, dx + \int_0^\infty y(x)\, dx = \lim_{a \to -\infty} \int_a^0 y(x)\, dx + \lim_{b \to \infty} \int_0^b y(x)\, dx.$$

The bell-shaped curve $y = e^{-x^2}$ covers a finite area (exactly $\sqrt{\pi}$). The region extends to infinity in both directions, and the separate areas are $\frac{1}{2}\sqrt{\pi}$. But notice:

$$\int_{-\infty}^\infty x\, dx \text{ is } \textbf{\textit{not defined}} \text{ even though } \int_{-b}^{b} x\, dx = 0 \textbf{\textit{ for every }} b.$$

The area under $y = x$ is $+\infty$ on one side of zero. The area is $-\infty$ on the other side. **We cannot accept** $\infty - \infty = 0$. The two areas must be separately finite, and in this case they are not.

EXAMPLE 9 $1/x$ has balancing regions left and right of $x = 0$. Compute $\int_{-1}^{1} dx/x$.

This integral does not exist. There is no answer, even for the region in Figure 7.8c. (They are mirror images because $1/x$ is an odd function.) You may feel that the combined integral from -1 to 1 should be zero. Cauchy agreed with that—his "principal value integral" is zero. But the rules say no: $\infty - \infty$ is not zero.

7.5 EXERCISES

Read-through questions

An improper integral $\int_a^b y(x)\,dx$ has lower limit $a = \underline{\quad a \quad}$ or upper limit $b = \underline{\quad b \quad}$ or y becomes $\underline{\quad c \quad}$ in the interval $a \leqslant x \leqslant b$. The example $\int_1^{\infty} dx/x^3$ is improper because $\underline{\quad d \quad}$. We should study the limit of $\int_1^b dx/x^3$ as $\underline{\quad e \quad}$. In practice we work directly with $-\frac{1}{2}x^{-2}]_1^{\infty} = \underline{\quad f \quad}$. For $p > 1$ the improper integral $\underline{\quad g \quad}$ is finite. For $p < 1$ the improper integral $\underline{\quad h \quad}$ is finite. For $y = e^{-x}$ the integral from 0 to ∞ is $\underline{\quad i \quad}$.

Suppose $0 \leqslant u(x) \leqslant v(x)$ for all x. The convergence of $\underline{\quad j \quad}$ implies the convergence of $\underline{\quad k \quad}$. The divergence of $\int u(x)\,dx$ $\underline{\quad l \quad}$ the divergence of $\int v(x)\,dx$. From $-\infty$ to ∞, the integral of $1/(e^x + e^{-x})$ converges by comparison with $\underline{\quad m \quad}$. Strictly speaking we split $(-\infty, \infty)$ into $(\underline{\quad n \quad}, 0)$ and $(0, \underline{\quad o \quad})$. Changing to $1/(e^x - e^{-x})$ gives divergence, because $\underline{\quad p \quad}$. Also $\int_{-\pi}^{\pi} dx/\sin x$ diverges by comparison with $\underline{\quad q \quad}$. The regions left and right of zero don't cancel because $\infty - \infty$ is $\underline{\quad r \quad}$.

Decide convergence or divergence in 1–16. Compute the integrals that converge.

1. $\int_1^{\infty} \dfrac{dx}{x^e}$

2. $\int_0^1 \dfrac{dx}{x^{\pi}}$

3. $\int_0^1 \dfrac{dx}{\sqrt{1-x}}$

4. $\int_0^1 \dfrac{dx}{1-x}$

5. $\int_{-\infty}^0 \dfrac{dx}{x^2+1}$

6. $\int_{-1}^1 \dfrac{dx}{\sqrt{1-x^2}}$

7. $\int_0^1 \dfrac{\ln x}{x}\,dx$

8. $\int_{-\infty}^{\infty} \sin x\,dx$

9. $\int_0^e \ln x\,dx$ (by parts)

10. $\int_0^{\infty} xe^{-x}\,dx$ (by parts)

11. $\int_{100}^{\infty} \dfrac{dx}{x(\ln x)(\ln \ln x)}$

12. $\int_{-\infty}^{\infty} \dfrac{x\,dx}{(x^2-1)^2}$

13. $\int_0^{\infty} \cos^2 x\,dx$

14. $\int_0^{\pi/2} \tan x\,dx$

15. $\int_0^{\infty} \dfrac{dx}{x^p}$

16. $\int_0^{\infty} \dfrac{e^x\,dx}{(e^x-1)^p}$

In 17–26, find a larger integral that converges or a smaller integral that diverges.

17. $\int_1^{\infty} \dfrac{dx}{x^6+1}$

18. $\int_0^1 \dfrac{dx}{x^6+1}$

19. $\int_0^{\infty} \dfrac{\sqrt{x}\,dx}{x^2+1}$

20. $\int_0^1 \dfrac{e^{-x}dx}{1-x}$

21. $\int_1^{\infty} e^{-x}\sin x\,dx$

22. $\int_1^{\infty} x^{-x}\,dx$

23. $\int_0^{\infty} e^{2x}e^{-x^2}\,dx$

24. $\int_0^1 \sqrt{-\ln x}\,dx$

25. $\int_0^{\infty} \dfrac{\sin^2 x}{x^2}\,dx$

26. $\int_1^{\infty} \left(\dfrac{1}{x} - \dfrac{1}{1+x}\right)dx$

27. If $p > 0$, integrate by parts to show that

$$\int_0^r x^p e^{-x}\,dx = p\int_0^r x^{p-1}e^{-x}\,dx.$$

The first integral is the definition of $p!$ So the equation is $p! = \underline{\qquad}$. In particular $0! = \underline{\qquad}$. Another notation for $p!$ is $\Gamma(p+1)$ using the *gamma function* emphasizes that p need not be an integer.

28. Compute $(-\frac{1}{2})!$ by substituting $x = u^2$:

$$\int_0^{\infty} x^{-1/2}e^{-x}\,dx = \underline{\qquad} = \sqrt{\pi} \text{ (known)}.$$

Then apply Problem 27 to find $(\frac{1}{2})!$

29. Integrate $\int_0^x x^2 e^{-x^2}\,dx$ by parts.

30. The *beta function* $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is finite when m and n are greater than $\underline{\qquad}$.

31. A *perpetual annuity* pays s dollars a year forever. With continuous interest rate c, its present value is $y_0 = \int_0^{\infty} se^{-ct}dt$. To receive \$1000/year at $c = 10\%$, you deposit $y_0 = \underline{\qquad}$.

32. In a perpetual annuity that pays once a year, the present value is $y_0 = s/a + s/a^2 + \cdots = \underline{\qquad}$. To receive \$1000/year at 10\% (now $a = 1.1$) you again deposit $y_0 = \underline{\qquad}$. Infinite sums are like improper integrals.

33. The work to move a satellite (mass m) infinitely far from the Earth (radius R, mass M) is $W = \int_R^{\infty} GMm\,dx/x^2$. Evaluate W. What *escape velocity* at liftoff gives an energy $\frac{1}{2}mv_0^2$ that equals W?

34 The escape velocity for a black hole exceeds the speed of light: $v_0 > 3 \cdot 10^8$ m/sec. The Earth has $GM = 4 \cdot 10^{14} \mathrm{m}^3/\mathrm{sec}^2$. If it were compressed to radius $R = \underline{\hspace{1cm}}$, the Earth would be a black hole.

35 Show how the area under $y = 1/2^x$ can be covered (draw a graph) by rectangles of area $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$. What is the exact area from $x = 0$ to $x = \infty$?

36 Explain this paradox:

$$\int_{-b}^{b} \frac{x\, dx}{1 + x^2} = 0 \text{ for every } b \text{ but } \int_{-\infty}^{\infty} \frac{x\, dx}{1 + x^2} \text{ diverges.}$$

37 Compute the area between $y = \sec x$ and $y = \tan x$ for $0 \leqslant x \leqslant \pi/2$. What is improper?

*38 Compute any of these integrals found by geniuses:

$$\int \frac{x^{-1/2} dx}{1 + x} = \pi \qquad \int_0^\infty \frac{e^{-x} - e^{-2x}}{x} dx = \ln 2$$

$$\int_0^\infty xe^{-x} \cos x\, dx = 0 \qquad \int_0^\infty \cos x^2 dx = \sqrt{\pi/8}.$$

39 For which p is $\displaystyle\int_0^\infty \frac{dx}{x^p + x^{-p}} = \infty$?

40 Explain from Figure 7.6c why the red area is 2, when Figure 7.6a has red area 1.

Resource: Calculus Online Textbook
Gilbert Strang